MATH 507a **QUALIFYING EXAM** **September 25, 2008**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway.

1.) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of random variables that converge to X in distribution. Assume that $P\{X_n \geq 1\} = 1$ for all n, and that $EX_n \to c < \infty$, as $n \to \infty$. Does it follow that $E\{\ln X_n\} \to E\{\ln X\}$? Justify your answer.

2.) Let $X_n, n \geq 1$ be iid Poisson random variables with parameter $\lambda > 0$. Show that

$$\limsup_{n\to\infty} \frac{X_n \ \ln(\ln n)}{\ln n} = 1$$

with probability 1.

3.) Assume that $X, X_1, X_2, \ldots$ are iid, with characteristic function $\phi(t) = \mathbb{E}\, e^{itX}$, and let $S_n = X_1 + \cdots + X_n$.

a) For a random variable X, what special property of its characteristic function ϕ holds if and only if X and $-X$ have the same distribution? (Show both implications.)

b) Express the characteristic function of the sample average, $\phi_{S_n/n}(t)$, in terms of ϕ.

c) Assume that $\phi'(0) = 0$. Show that S_n/n converges to zero in probability. (The converse is also true, but you are NOT being asked to show this.) [Hints: Show that $\phi(u) = 1 + o(u)$ as $u \to 0$. Use $\log(1+x)$ is asymptotic to x for small x, to show that for each fixed t, $(1+o(t/n))^n \to 1$ as $n \to \infty$. State how this implies the desired convergence.]

d) and e): Assume that X has density

$$f(x) = c\frac{1}{x^2 \ \ln|x|}$$

for $|x| > 4$, and $f(x) = 0$ for $-4 \leq x \leq 4$, where c is the appropriate normalizing constant.

d) Show that $\mathbb{E}\,|X| = \infty$.

e) Show that the characteristic function for X has $\phi'(0) = 0$. [Hints: Use part a). Express $1 - \phi(t)$ as an integral over $x > 4$, and use the change of variables $y = tx$ to show that $|1 - \phi(t)|/t \to 0$ as $t \to 0$. You might use $|1 - \cos y| \leq y^2$ for all y, together with dominated convergence.]