MATH 507a GRADUATE EXAM

SPRING 2008

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) For $\epsilon > 0$ let $\{X_i^\epsilon\}$ be i.i.d. with $P(X_i^\epsilon = \epsilon) = P(X_i^\epsilon = -\epsilon) = 1/2$. Let N_ϵ have a Poisson distribution with parameter λ/ϵ^2, independent of the X_i^ϵ's. Let

$$S_\epsilon = \sum_{i=1}^{N_\epsilon} X_i^\epsilon.$$

(a) Find the characteristic function φ_ϵ of S_ϵ.
(b) Find $\lim_{\epsilon\to 0} \varphi_\epsilon(t)$. What does this tell you about the random variables S_ϵ?

(2) Suppose $X_n \to X$ in distribution and $Y_n \to 0$ in distribution. Show that $X_n + Y_n \to X$ in distribution.

(3) Let U_1, U_2, ..., be i.i.d. sequence of Gaussian random variables with the common distribution $\mathcal{N}(0,1)$. Let a_0, a_1, a_2, ... be the real numbers such that $a_j a_{j+1} = 0$ for all $j \geq 0$ and that the series Σa_n^2 converge. Define

$$V_n = \sum_{k=1}^{n} a_{n-i} U_i, \qquad n = 1, 2, \cdots .$$

(a) Show that V_n and V_{n+1} are independent for all $n \geq 1$.
(b) Show that with probability 1,

$$\limsup_{n\to\infty} \frac{V_n}{\sqrt{\ln n}} \leq \sqrt{2 \sum_{j=0}^{\infty} a_j^2}.$$

HINT: You can take as given the inequality $P(U_1 \geq x) \leq \frac{1}{\sqrt{2\pi}x} e^{-x^2/2}$, for $x > 0$.

(4) Prove the following inequality, sometimes known as Cantelli's inequality, and sometimes called the one-sided Chebyshev inequality: If X has mean 0 and variance 1, then for any $c \geq 0$,

$$P(X \geq c) \leq \frac{1}{1 + c^2}.$$

HINT: Relate the event $\{X \geq c\}$ to $\{(X + t)^2 \geq (c + t)^2\}$, for appropriate t.