MATH 507a GRADUATE EXAM

FALL 2007

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) In a sequence $X_0, X_1, X_2, \ldots$ of coin tosses, the length L_n of the head run starting at time n is defined by $\{L_n \geq k\} = \{1 = X_n = X_{n+1} = \cdots = X_{n+k-1}\}$. Consider fair coin tossing, so that $P(L_n \geq k) = 1/2^k$. With all logs taken base 2, show that $P(L_n > \log n + \theta \log \log n$ infinitely often$) = 0$ whenever $\theta > 1$.

(2) Suppose X_n, X_∞ are r.v.'s with characteristic functions ϕ_n, ϕ_∞, all dominated by a function g in L^1 (that is, $|\phi_n(t)| \leq g(t)$ for all n and all t.) If $\phi_n \to \phi_\infty$ pointwise, show that X_n and X_∞ have densities, call them f_n and f_∞, and $f_n \to f_\infty$ uniformly.

(3) Suppose $X_n, n \geq 1$, are r.v.'s with d.f.'s F_n satisfying $EX_n^2 < \infty$ for all n, and

$$\lim_{A\to\infty} \sup_n \frac{\int_{\{x:|x|>A\}} x^2 \, dF_n(x)}{\int_{\mathbb{R}} x^2 \, dF_n(x)} = 0.$$

Show that $\{F_n\}$ is tight. HINT: $\int_{\mathbb{R}} = \int_{\{x:|x|\leq A\}} + \int_{\{x:|x|>A\}}$.

(4)(a) Let $\varphi \geq 0$ be a nondecreasing function on $\mathbb{R}$. Show that for every random variable Y and $t \in \mathbb{R}$,

$$P(Y > t) \leq \frac{E\varphi(Y)}{\varphi(t)}.$$

(b) Let $X_1, X_2, \ldots$ i.i.d variables, with $M(\lambda) := E\left[e^{\lambda X_1}\right] < \infty$ for every $\lambda \in \mathbb{R}$, and $E[X_1] = 0$. Let $S_n = X_1 + \cdots + X_n$. Show, that for every $x > 0$ and $n \geq 1$

$$\frac{1}{n} \log P\left(S_n > nx\right) \leq -I(x),$$

with $I(x) = \sup_{\lambda>0}\left[\lambda x - \log M(\lambda)\right]$. HINT: Use (a).