Screening Exam on Numerical Analysis – Spring 2008

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1. **Linear Equations**

   a. Let $A \in \mathbb{R}^{n\times n}$. Show that the sum of eigenvalues of A is equal to the sum of the diagonal elements of A.

   Hint: compare $det(A - \lambda I)$ and $p(\lambda) = (\lambda_1 - \lambda)...(\lambda_n - \lambda)$.

   b. Consider solving $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is non-singular. Derive the estimate for the relative error in the solution x if the right hand side b is perturbed by δ.

   c. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1+\epsilon \end{pmatrix}, \epsilon > 0.$

   How small ϵ should be for you to call the matrix ill-conditioned?

2. **Least Squares**

   a. Prove the following Theorem:

   If $A = QR$ with $Q^TQ = I$, then the least squares solution to $Ax = b$ is $x = R^{-1}Q^Tb$,

   where A is a $n \times m$ matrix.

   b. Compute QR factorization of $A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 4 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

3. **Eigenvalue Problems**

   a. Let $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{n\times m}$ be two given matrices. Prove that the nonzero eigenvalues of BC and CB are the same.

   b. Suppose $n \geq 1$ and $T \in \mathbb{R}^{(2n+1)\times(2n+1)}$ is a tridiagonal matrix of the form

$$T = \begin{bmatrix} a_1 & b_2 & & & \\ b_2 & a_2 & b_3 & & \\ & b_3 & \ddots & \ddots & \\ & & \ddots & a_{2n} & b_{2n+1} \\ & & & b_{2n+1} & a_{2n+1} \end{bmatrix}$$

   $b_i \neq 0$ for all i but $b_{n+1} = 0$. Show that there exists at least one eigenvalue with multiplicity one. (Hint: Study the multiplicity of eigenvalues of $T_1 \in \mathbb{R}^{n\times n}$ and $T_2 \in \mathbb{R}^{(n+1)\times(n+1)}$, where $T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}$.)

   c. Perform one step of a QR algorithm without shift on $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & 0 \end{bmatrix}$

4. **Iterative Methods**

Consider $A \in \mathbb{R}^{n \times n}$ to be strictly positive definite, that is, $\xi^T A \xi > 0$ for all nonzero $\xi \in \mathbb{R}^n$. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ are eigenvalues and spectral radius is $\rho = max_i |\lambda_i(A)|$.

a. Show that $\rho \leq \|A\|$ for any consistent matrix norm $\|\cdot\|$.

b. We use the Richardson iteration formula to solve $Ax = b$:

$$x^{(k+1)} = x^{(k)} + \omega(b - Ax^{(k)}), \quad k = 0, 1, 2, \ldots \tag{1}$$

Show that (1) is convergent for any $0 < \omega < 2/\rho$.

c. Compare two iterations of (1), by taking $\omega_1 = 1/\rho$ and $\omega_2 = 1/(\lambda_1 + \lambda_n)$. Which one has faster convergence rate? (Hint: compare spectral of the iteration matrix for these two cases).