Screening Exam in Numerical Analysis – Fall 2007

*Name*________________________________

Solve any three out of the following four problems (do not attempt more than three).

1. **linear equations**

   a) Define the condition number of a matrix A and explain briefly how it is related to the numerical solving of the system $Ax = b$. Prove that $\|A\| \cdot \|A^{-1}\| \geq 1$ for any operator norm $\| \cdot \|$, provided that A^{-1} exists.

   b) Let $A \in R^{n\times n}$ be a nonsingular matrix. Show that the condition number $\kappa(A)$ satisfies
   $$\kappa(A) \geq \frac{\|A\|}{\|B - A\|}$$
   for any singular matrix $B \in R^{n\times n}$.

   c) Suppose $0 < |\varepsilon| < 1$. Given matrix
   $$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon \end{bmatrix}.$$
   Show that
   $$\kappa_\infty(A) \geq \frac{3}{2\varepsilon},$$
   where κ_∞ is the condition number with respect to infinity norm.

2. **iterative methods**

   a) Let $B \in R^{n\times n}$ and $c \in R^n$. Denote the spectral radius by $\rho(B)$. Suppose $\rho(B) < 1$, then show that
   $$\lim_{n\to\infty} B^n = 0.$$
   b) Prove that A stationary iterative method
   $$x^{(n+1)} = Bx^{(n)} + c$$
   is convergent to the unique solution of $(I - B)x = c$ for all initial vector $x^{(0)}$ provided that $\rho(B) < 1$.

   c) Consider the iteration of (b). Let $\|B\| \leq \beta < 1$, and $\|x^{(k)} - x^{(k-1)}\| \leq \varepsilon$ for some k. Prove that
   $$\|x - x^{(k)}\| \leq \frac{\beta\varepsilon}{1 - \beta}$$

3. eigenproblems

a. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 1 & 9.8 \end{pmatrix}$.

i. Explain the slow rate of convergence of the power method with A.

ii. Choose a suitable shift σ so that power method converges to the largest eigenvalue of A. Explain the improvement of the rate of convergence with $A - \sigma I$.

b. Show that a product (both HR and RH) of a nonsingular right triangular matrix (R) with a Hessenberg matrix (H) is a Hessenberg matrix.

4. least squares

Consider the following least squares minimization problem: find a minimum norm solution of the functional

$$J(x) = \|Bx - c\|^2,$$

where B is a $n \times m$ real matrix and $c \in \mathbb{R}^m$ is given.

(a) Write down the minimum norm solution of the least square minimization problem as a solution of system of linear equation of the form $Ax = y$ and determine whether or not a unique solution of $Ax = y$ exists.

(b) Express the minimum norm solution of the least square minimization problem as a function of the eigenvectors of the matrix A. Justify your answer.

A penalty method approach for solving this problem is to consider an alternative minimization problem of finding x_λ that minimizes

$$J_\lambda(x) = \|Bx - c\|^2 + \lambda\|x\|^2,$$

for $\lambda > 0$.

(c) Write down the solution of penalty method as a function of the eigenvectors of the matrix A.

(d) Show that x_λ tends toward the solution x of the original least square minimization problem as λ tends toward infinity.