Screening Exam in Numerical Analysis – Fall 2008

Linear Algebra

1. Perform LU factorization on Hilbert matrix $H_3 = [h_{ij}]_{1\le i,j\le 3}$, with elements
$$h_{ij} = \frac{1}{i+j-1}.$$

2. Let $A \in \mathbb{R}^{n\times n}$ have LU factorization, and $P \in \mathbb{R}^{n\times n}$ be given by $P = (e_n, e_{n-1}, \ldots, e_1)$, where e_i is unit vector. Prove that PAP has UL factorization, that is, there exists upper triangular U and lower triangular L satisfying $PAP = UL$.

3. Let $B = [b_{ij}]_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ be symmetric positive definite. Show that for any $1 \le i, j, k \le n$
$$b_{ij} + b_{jk} + b_{ki} \le b_{ii} + b_{jj} + b_{kk}.$$

Least squares

1. Let $A = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find orthonormal matrix $Q \in \mathbb{R}^{3\times 2}$ and upper-triangle matrix $R \in \mathbb{R}^{2\times 2}$, such that $A = QR$.

2. Let $b = (2, -1, 1)^T$. Find $x \in \mathbb{R}^{2\times 1}$, which minimizes $\|Ax - b\|_2$.

3. Prove Hadamard's determinant inequality:
If $A = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^{n\times n}$, then
$$|\det(A)| \le \Pi_{j=1}^{n} \|a_j\|_2,$$
with equality only if A^TA is diagonal matrix or A has a zero column. (Hint: Consider QR factorization $A = QR$.)

Iterative Methods

1. Consider solving $Au = f$, where $A \in R^{n \times n}$ is consistently ordered.

   a. Give the matrix form of Jacobi, Gauss-Seidel and SOR iterations.

   b. If the eigenvalues of the Jacobi iteration matrix, Q_J are $\lambda_i(Q_J) = cos(\frac{\pi i}{n+1})$, i=1,...,n, what is the optimal over-relaxation parameter ω_{opt}?

2. Consider solving $Au = f$, where $A \in R^{n \times n}$ and $A = A^T$.

   a. Define the conjugate gradient method.

   b. Give the estimate of its rate of convergence.

   c. Compute estimate of the rate of convergence if the eigenvalues of A are $\lambda_i(A) = 2 + 2cos(\frac{\pi i}{n+1})$, i=1,...,n.

Eigenvalue Problems.

1. Show that if X is a unitary matrix, and the first column of X is an eigenvector of A associated with eigenvalue λ, then

$$X^*AX = \begin{bmatrix} \lambda & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$

2. Consider the matrix

$$A = \begin{bmatrix} -2 & 1 & 1 \\ -2 & 2 & 1 \\ 2 & -2 & 3 \end{bmatrix},$$

   with an eigenvalue $\lambda = 2$ and corresponding eigenvector $x = [1, 2, 2]^T$.

   Construct a Householder matrix H such that

$$HAH^* = \begin{bmatrix} 2 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$