Geometry/Topology Qualifying Exam

Fall 2008

Solve all **SIX** *problems. Partial credit will be given to partial solutions.*

1. Consider the map $d_f : \Omega^i(M) \to \Omega^{i+1}(M)$ given by $\omega \mapsto d\omega + df \wedge \omega$, where M is a smooth manifold, $\Omega^i(M)$ is the set of smooth i-forms on M, and f is a smooth function on M.

   (a) Show that d_f is a cochain map, i.e., $d_f \circ d_f = 0$.

   (b) Let $H^i_f(M)$ be the ith cohomology group of the cochain complex $(\Omega^i(M), d_f)$. Show that $H^0_f(M) \cong \mathbb{R}$ when M is the real line $\mathbb{R}$.

2. Show that, when m, $n > 0$, the homomorphism $f^* : H^k_{dR}(S^m \times S^n) \to H^k_{dR}(S^{m+n})$ induced in de Rham cohomology by $f : S^{m+n} \to S^m \times S^n$ is trivial for all $k > 0$. Here S^n is the n-dimensional sphere. [Possible hint: Construct a volume form for $S^m \times S^n$ from a volume form on S^m and a volume form on S^n.]

3. Prove that the set $C = \{(x, y) \mid y^2 - x^3 = 0\}$ is not a smooth submanifold of the plane. [Hint: What is the space of tangent vectors in $T_{(0,0)}\mathbb{R}^2$ which are tangent to C?]

4. Let T be the surface obtained by revolving the circle $\{(x, y, z) \mid z = 0, (x - R)^2 + y^2 = r^2\}$ around the y-axis, where $R > r$. Compute the integral
$$\int_T xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

5. Let B^3 be the (closed) 3-dimensional ball, and let K be a closed, connected 1-dimensional submanifold of B^3 with $\partial K = K \cap \partial B^3 = 2$ points. Compute the homology of the complement $B^3 - K$ (= an apple minus a wormhole).

6. Recall that two covering spaces $p : \widetilde{X} \to X$ and $p' : \widetilde{X}' \to X$ are *isomorphic* if there exists a homeomorphism $\widetilde{\phi} : \widetilde{X} \xrightarrow{\sim} \widetilde{X}'$ such that $p' \circ \widetilde{\phi} = p$. Consider the covering spaces $p : \widetilde{X} \to X$ of the torus $X = S^1 \times S^1$ whose fiber $p^{-1}(x_0)$ at any point $x_0 \in X$ consists of 3 points. How many distinct isomorphism classes of such coverings are there?