Geometry and Topology Graduate Exam
February 2008

1. Let $p : \widetilde{X} \to X$ be a covering with path connected base X, and let G be its automorphism group, consisting of those homeomorphisms $\varphi : \widetilde{X} \to \widetilde{X}$ such that $p \circ \varphi = \varphi$. Pick base points $x_0 \in X$ and $\widetilde{x}_0 \in \widetilde{X}$ with $p(\widetilde{x}_0) = x_0$. Suppose that, for any two $\widetilde{x}_0'$, $\widetilde{x}_0'' \in p^{-1}(x_0)$, there exists $\varphi \in G$ such that $\varphi(\widetilde{x}_0'') = \widetilde{x}_0'$. Show that there is an exact sequence

$$1 \longrightarrow \pi_1(\widetilde{X}; \widetilde{x}_0) \xrightarrow{p_*} \pi_1(X; x_0) \longrightarrow G \longrightarrow 1.$$

2. Consider on $\mathbb{R}^n$ the standard inner product $(\vec{a}, \vec{b}) = \sum_{i=1}^n a_i b_i$, when $\vec{a} = (a_1, a_2, \dots a_n)$ and $\vec{b} = (b_1, b_2, \dots, b_n)$. Let V be a vector subspace of $\mathbb{R}^n$, and let $\pi : \mathbb{R}^n \to V$ be the orthogonal projection with respect to the above inner product. If M is a submanifold of $\mathbb{R}^n$, show that the restriction $\pi_{|M} : M \to V$ is an immersion if and only if $T_x M \cap V^\perp = \{0\}$ for every $x \in M$.

3. Let $f : X \to X$ be a map homotopic to a constant map, and let $M_f = X \times [0,1] / \sim$ where the equivalence relation $\sim$ identifies $(x, 0)$ to $(f(x), 1)$. Compute the homology groups of M_f.

4. Consider a differentiable map $f : S^{2n-1} \to S^n$, with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$ is a differential form of degree n on S^n such that $\int_{S^n} \alpha = 1$, let $f^*(\alpha) \in \Omega^n(S^{2n-1})$ be its pull-back under the map f.

a) Show that there exists $\beta \in \Omega^{n-1}(S^{2n-1})$ such that $f^*(\alpha) = d\beta$.

b) Show that the integral $I(f) = \int_{S^{2n-1}} \beta \wedge d\beta$ is independent of the choice of β and α. It may be useful to remember that the map $H^n(S^n) \to \mathbb{R}$ defined by $\gamma \mapsto \int_{S^n} \gamma$ is an isomorphism.

5. Let $\omega \in \Omega^2(S^2)$ be the restriction of the 2–form

$$x\, dy \wedge dz + z\, dx \wedge dy + y\, dz \wedge dx$$

to the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\}$. Compute the integral $\int_{S^2} \omega$.

6. Recall that the 1–dimensional projective space $\mathbb{RP}^1$ consists of all lines in $\mathbb{R}^2$ passing through the origin. Let $f : \mathbb{R} \to \mathbb{RP}^1$ associate to $x \in \mathbb{R}$ the line passing through $(x, 1)$ and the origin. Finally, let $P(x)$ be a polynomial function of the variable x.

a) Show that there is no differential form ω on $\mathbb{RP}^1$ such that $f^*(\omega) = P(x)\, dx$.

b) Show that there exists a vector field V on $\mathbb{RP}^1$ such that $f^*(V) = P(x) \frac{\partial}{\partial x}$ if and only if the degree of $P(x)$ is ≤ 2.

7. Let M be a compact differentiable manifold, and let $C^\infty(M)$ be the algebra of all differentiable functions $M \to \mathbb{R}$. Let $\mathcal{I}$ be a maximal ideal of $C^\infty(M)$. Show that there is a point $x_0 \in M$ such that $\mathcal{I} = \{f \in C^\infty(M); f(x_0) = 0\}$. (Possible hint: Suppose that the property is not true and show that, for every $x \in M$, there exists a non-negative function $f \in \mathcal{I}$ such that $f(x) > 0$.)