Graduate Exam
Geometry and Topology
Fall 2007

Problem 1. Let X be a path connected space such that $H_p(X, \mathbb{Z}) = 0$ for every p with $0 < p \leq n$. If $X \times S^n$ denotes the product of X with the n–dimensional sphere S^n, compute the homology groups $H_p(X \times S^n; \mathbb{Z})$ for every p with $0 < p \leq n$.

Problem 2. Let C_1 and C_2 be two disjoint circles in $\mathbb{R}^3$, and let $A = S^1 \times [0, 1]$ denote the cylinder. Let X be the space obtained from the disjoint union $X \sqcup A$ by gluing the boundary component $S^1 \times \{0\}$ of A to the circle C_1 by a homeomorphism, and by gluing the other boundary component $S^1 \times \{1\}$ to C_2 by another homeomorphism. Compute the fundamental group of the space X so obtained.

Problem 3. Let $M_n(\mathbb{R})$ be the vector space of $n \times n$ matrices with coefficients in $\mathbb{R}$, and consider the determinant function $det : M_n(\mathbb{R}) \to \mathbb{R}$, which to a matrix A associates its determinant $det(A)$. Compute the differential map (also called tangent map) of the function det at the identity matrix $I_n \in M_n(\mathbb{R})$.

Problem 4. Let M be a compact orientable n–dimensional manifold whose boundary ∂M is homeomorphism to the sphere $S^{n-1} \subset \mathbb{R}^n$ by a homeomorphism $f : \partial M \to S^{n-1}$. Let F be a continuous map $F : M \to \mathbb{R}^n$ whose restriction to the boundary ∂M coincides with f. Show that the image $F(M)$ necessarily contains the center O of the sphere S^{n-1}.

Problem 5. Let Ω be the open shell in $\mathbb{R}^2$ consisting of those $(x, y) \in \mathbb{R}^2$ such that $1 < x^2 + y^2 < 10$, and consider the 1–form

$$\omega = \frac{x\,dy - y\,dx}{4x^2 + y^2}$$

a) Show that ω is closed in Ω.

b) Show that ω is not closed in Ω. (Possible hint: consider an ellipse of equation $4x^2 + y^2 =$ constant).

Problem 6. Let $\mathbb{RP}^2$ denote the real projective plane of dimension 2. Consider the map $\varphi : \mathbb{R}^2 \to \mathbb{RP}^2$ which to $(x, y) \in \mathbb{R}^2$ associates the element of $\mathbb{RP}^2$ represented by the line passing through the point $(x, y, 1)$. (Recall that $\mathbb{RP}^2$ is the space of lines passing through the origin in $\mathbb{R}^3$.) If $C = \{(x, y) \in \mathbb{R}^2; y^2 = x^3 - x\}$, show that the closure $\overline{\varphi(C)}$ of $\varphi(C)$ in $\mathbb{RP}^2$ is a differentiable submanifold of $\mathbb{RP}^2$.

Problem 7. Let M and N be two compact connected manifolds of the same dimension n, and let $f : M \to N$ be a continuous map. Suppose that the homomorphism $H_n(f) : H_n(M; \mathbb{Z}) \to H_n(\mathbb{Z})$ induced by f is not 0. If $f_* : \pi_1(M, x_0) \to \pi_1(N, f(x_0))$ is the homomorphism induced by f between the fundamental groups, show that its image $f_*(\pi(M; x_0))$ has finite index in $\pi(N; f(x_0))$. (Possible hint: Consider a suitable covering of N.)