1. Let $\{P_\theta, \theta \in \Theta\}$ be a family of probability distributions. A statistic $V$ is called *ancillary* for $\theta$ if its distribution does not depend on $\theta$.

   (a) Let $X_1, \ldots, X_n$ have normal distribution $N(\mu, 1)$. Show that $V = X_1 - \bar{X}$ is an ancillary statistic for $\mu$.

   (b) Prove that if $T$ is a complete sufficient statistic for the family $\{P_\theta, \theta \in \Theta\}$, then any ancillary statistic $V$ is independent of $T$. (This is a theorem due to Basu).
   *(Hint: show that for any (measurable) set A,*

   $$P_\theta(V \in A|T = t) = P(V \in A|T = t) = P(V \in A),$$

   *and derive the conclusion).*

2. With $\theta > 0$ unknown, let a sample consist of $X_1, \ldots, X_n$, independent observations with distribution

   $$F(y;\theta) = 1 - \sqrt{1 - \frac{y}{\theta}}, \quad 0 < y < \theta.$$

   (a) Prove that the maximum likelihood estimate of $\theta$ based on the sample is the maximum order statistic

   $$X_{(n)} = \max_{1 \le i \le n} X_i.$$

   (b) Determine a sequence of positive numbers $a_n$ and a non-trivial distribution for a randon variable $X$ such that

   $$a_n(\theta - X_{(n)}) \to_d X.$$

   (c) Compare the rate $a_n$ in part b) to the rate of a parametric estimation problem whose regularity would allow the application of the Cramer Rao bound. Comment.