REAL ANALYSIS GRADUATE EXAM

Fall 2016

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{f_n\}_{n=1}^{\infty}$ be a sequence of *nonnegative* measurable functions. Prove that $f_n \to 0$ in measure if and only if

$$\lim_n \int \frac{f_n}{f_n + 1} d\mu = 0.$$

2. Let $(X, \mathcal{F}, \mu)$ be a *finite* measure space, and let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ be a sequence of sets. Assume that $\mu(A_n) \geq \delta$ for all $n \in \mathbb{N}$, where $\delta > 0$. Prove that there exists a set $S \in \mathcal{F}$ of positive measure such that every $x \in S$ is in A_j for infinitely many j.

3. Let $f_n \colon [0, 1] \to [0, \infty)$ be Lebesgue measurable and such that $f_n(x) \to 0$ for almost every x. Assume that

$$\sup_n \int_0^1 \phi(f_n(x))\, dx \leq 1$$

for some continuous $\phi \colon [0, \infty) \to [0, \infty)$ which satisfies $\phi(t)/t \to \infty$ as $t \to \infty$. Prove that $\int_0^1 f_n(x)\, dx \to 0$ as $n \to \infty$. (Provide a detailed proof.)

4. Let $h \colon [0, \infty) \to \mathbb{R}$ be continuous with compact support. Prove that

$$\lim_{\epsilon \to 0+} \int_{\epsilon}^{\infty} \frac{h(\alpha x) - h(\beta x)}{x}\, dx = h(0) \log \frac{\alpha}{\beta}$$

for every $\alpha, \beta > 0$.