Algebra Qualifying Exam - Fall 2016

1. If $R := \mathbb{C}[x,y]/(y^2 - x^3 - 1)$, then describe all the maximal ideals in R.
2. Suppose F is a field, and $\mathfrak{b}_n(F)$ is the F-algebra of upper-triangular matrices, i.e., the subalgebra of $M_n(F)$ consisting of matrices X such that $X_{ij} = 0$ when $i > j$. Describe the Jacobson radical of $\mathfrak{b}_n(F)$, the simple modules, and the maximal semi-simple quotient.
3. Let $\mathbb{F}_5$ be the finite field with 5 elements, and consider the group $G = PGL_2(\mathbb{F}_5)$ (i.e., the quotient of the group of invertible 2×2-matrices over $\mathbb{F}_5$ by the subgroup of scalar multiples of the identity).
   (a) What is the order of G?
   (b) Describe $N_G(P)$ where P is a Sylow-5 subgroup of G.
   (c) If $H \subset G$ is a subgroup, can H have order 15, 20 or 30?
4. Let A be an $n \times n$ matrix over $\mathbb{Z}$. Let V be the $\mathbb{Z}$-module of column vectors of size n over $\mathbb{Z}$.
   (a) Prove that the size of V/AV is equal to the absolute value of $\det(A)$ if $\det(A) \neq 0$.
   (b) Prove that V/AV is infinite if $\det(A) = 0$.

   (hint: use the theory of finitely generated modules $\mathbb{Z}$-modules)
5. Let V be a finite dimensional right module over a division ring D. Let W be a D-submodule of V.
   (a) Let $I(W) = \{f \in \mathrm{End}_D(V) | f(W) = 0\}$. Prove that $I(W)$ is a left ideal of $\mathrm{End}(V)$.
   (b) Prove that any left ideal of $\mathrm{End}_D(V)$ is $I(W)$ for some submodule W.
6. Let p and q be distinct primes. Let F be the subfield of $\mathbb{C}$ generated by the pq-roots of unity. Let a, b be squarefree integers all greater than 1. Let $c, d \in \mathbb{C}$ with $c^p = a$ and $d^q = b$. Let $K = F(c, d)$.
   (a) Show that $K/\mathbb{Q}$ is a Galois extension.
   (b) Describe the Galois group of K/F.
   (c) Show that any intermediate field $F \subset L \subset K$ satisfies $L = F(S)$ where S is some subset of $\{c, d\}$.