# MATH 507a GRADUATE EXAM

## Fall 2016

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Suppose $X_1, X_2, \dots$ are independent non-negative random variables. Show that $\sum_{n=1}^{\infty} X_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} E[X_n/(1+X_n)] < \infty$. HINT: Consider the truncated variables $X_n' = \min(X_n, 1)$.

(2) Recall that $m$ is a median of $X$ if $P(X \le m) \ge 1/2$ and $P(X \ge m) \ge 1/2$.

Suppose $X$ has density $f$ and $E|X| < \infty$. Show that $m$ is a median of $X$ if and only if the function $g(a) = E(|X-a|)$ is minimized (not necessarily uniquely) at $a = m$.

(3) Let $X_1, X_2, \dots$ be iid random variables with common density function $f$. Define $N_1, N_2, \dots$ to be the times when the values $X_i$ set a new record high, that is, $N_1 = 1, N_i = \inf\{k > N_{i-1} : X_k > X_{N_{i-1}}\}$. Let $Y_i = X_{N_{i+1}} - X_{N_i}$; this is the margin by which the record is set.

For each case (a) and (b) determine whether the sequence $\{Y_i\}$ converges in distribution as $i \to \infty$, and find the limiting distribution if it exists. HINT: Condition on $X_{N_i}$.

(a) $f(t) = e^{-t}$ for $t > 0$, $f(t) = 0$ for $t \le 0$.

(b) $f(t) = t^{-2}$ for $t > 1$, $f(t) = 0$ for $t \le 1$.