DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Fall 2008

1. Consider the linear p-periodic system in $\mathbb{R}^n$ with A continuous,

$$x'(t) = A(t)x(t), \qquad A(t+p) = A(t).$$

   (a) State Floquet's Theorem,

   (b) Assume each eigenvalue, $\lambda(t)$ of $A(t)$ satisfies

$$\mathcal{R}e\lambda(t) \leq -1.$$

   What can be concluded about the asymptotic stability of the stationary solution, $x(t) \equiv 0$?

   (c) Describe with a sketch the skew-product dynamical system defined by the above system.

2. Show that if real valued continuos functions $f(x)$, $g(x)$, and $h(x)$ satisfy the inequalities

$$f(x) \geq 0, \quad g(x) \leq h(x) + \int_0^x f(\xi)g(\xi)\,d\xi \tag{1}$$

on an interval $0 \leq x \leq x_0$, then

$$g(x) \leq h(x) + \int_0^x \left\{ f(\zeta)h(\xi) \exp\Big[\int_\xi^x f(\eta)\,d\eta\Big] \right\} d\xi$$

on $0 \leq x \leq x_0$.

Hint: If we put

$$y(x) = \int_0^x f(\xi)g(\xi)\,d\xi,$$

then $\dfrac{dy}{dx} \leq f(x)h(x) + f(x)y$ and $y(0) = 0$.

3. Let $\alpha \in [-1, 1], \beta > 0$ and $z = x + iy$ a complex variable. Consider the system in $\mathbb{R}^2$,

$$z' = (\alpha + i\beta)z - |z|^4 z.$$

Describe in detail the change that takes place in the phase portrait as α varies from -1 to 1. Prove your statements.

4. Let $u(x,t)$ be a bounded solution to the Cauchy Problem for the Heat equation

$$u_t = a^2 u_{xx}, \quad t > 0, \quad x \in \mathbb{R}, \quad a > 0$$
$$u(x,0) = \varphi(x).$$

Here $\varphi(x) \in C(\mathbb{R})$ satisfies

$$\lim_{x\to\infty} \varphi(x) = b, \qquad \lim_{x\to-\infty} \varphi(x) = c$$

Compute the limit of $u(x,t)$ as t goes to infinity. Justify your argument carefully.

Hint: Use the explicit form of the solution, a change of variables $z = (y-x)/\sqrt{4a^2t}$ and a splitting of the integral into three parts.

5. Let $\varphi(x)$ be a function in $C_0^\infty(\mathbb{R}^3)$, and consider the following damped wave equation

$$u_{tt} - \Delta u + \varphi(x) u_t = 0, \quad (x,t) \in \mathbb{R}^3 \times \mathbb{R}$$
$$u|_{t=0} = f, \quad u_t|_{t=0} = g.$$

a) Fix $x_0 \in \mathbb{R}^3$ and $t_0 > 0$. Let $C = \{(x,t) : 0 \le t \le t_0,\ |x - x_0| \le |t - t_0|\}$ be the cone of dependence for the point (x_0, t_0) and define $B_\tau = C \cap \{t = \tau\}$. Prove that the energy

$$e(\tau) = \frac{1}{2} \int_{B_\tau} (|u_t|^2 + |\nabla u|^2)\, dt$$

is decreasing and that if $u = u_t = 0$ in B_0 then $u = 0$ identically on C.

b) Use the fact that $f,\ g \in C_0^\infty(\mathbb{R}^3)$ to conclude that the energy of the solution

$$E(t) = \frac{1}{2} \int_{\mathbb{R}^3} (|u_t|^2 + |\nabla u|^2)\, dt$$

is decreasing.

6. Find the general solution of the equation

$$x u_{xx} + u_{xy} = 0; \quad u = u(x; y)$$

7. (a) Let $U \subset \mathbb{R}^n$ be an open bounded domain. Let u; v be two harmonic functions in U, which are continuous in U. Show that if $u \le v$ on the boundary ∂U of U, then $u \le v$ in U.

(b) Consider the domain $D = \{x \in \mathbb{R}^n : |x| < 1\} \setminus \{0\}$, where $n \ge 2$. Suppose that u is a harmonic function in D, it is continuous in D, and $u = 0$ on the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$. Prove that $u \equiv 0$ in D.

Hint: Compare u with functions of the form $const \cdot \Phi(x)$ on domains of the form $\{x \in \mathbb{R}^n : r < |x| < 1\}$, where $\Phi(x)$ denotes the fundamental solution on $\mathbb{R}^n$.