DIFFERENTIAL EQUATIONS QUALIFYING EXAM–Spring 2008

1. a) Solve the linear partial differential equation

$$e^x u_x + u_y = u \quad \text{with } u(x,0) = g(x).$$

b) Solve the nonlinear partial differential equation

$$x^2 u_x + y^2 u_y = u^2 \quad \text{with } u = 1 \text{ on the line } y = 2x$$

2. Let $c \in \mathbb{R}$. Write down an explicit formula for a solution of

$$\begin{aligned} u_t - \Delta u + cu &= f \qquad && \text{in } \mathbb{R}^n \times (0, \infty) \\ u &= g && \text{on } \mathbb{R}^n \times t = 0. \end{aligned}$$

3. Let $B(0,1)$ be the unit ball in $\mathbb{R}^n$.

a) If

$$u(x) = |x|^{-\alpha}, \quad x \in B(0,1)$$

For what values of α, n, p the function u is in the Sobolev space $W^{1,p}(B(0,1))$.

b) If $u(x) = \ln\ln(1 + \frac{1}{|x|})$, for $x \in B(0,1)$, Prove that $u \in W^{1,n}(B(0,1)))$ but not in $L^\infty(B(0,1))$.

4. Let $g : R^2 \times (-1,1) \to R^2$ be of class C^4 and consider the mapping $x \to g(x,\mu)$ where $g(0,\mu) \equiv 0$ and $\frac{\partial g}{\partial x}(0,\mu)$ has complex eigenvalues $\lambda(\mu), \bar{\lambda}(\mu)$ that leave the unit circle as μ increases through 0, i.e. $|\lambda(0)| = 1$ and $\frac{d|\lambda(\mu)|}{d\mu} > 0$. After a linear transformation and letting $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, the mapping takes the form

$$z \to \lambda(\mu)z + \dots .$$

(a) State conditions on $\lambda(0)$ that allow the mapping to be transformed into the *Normal Form*

$$w \to we^{\alpha(\mu)+\beta(\mu)|w|^2} + \mathcal{O}(|w|^4). \tag{1}$$

(b) Give a condition on $\beta(0)$ that guarantees a Neimark-Sacker bifurcation of the origin into an asymptotically stable invariant curve $\Gamma(\mu)$ for $0 < \mu < \mu*$ surrounding the origin.

(c) Write down an expression for the first approximation to Γ. **Hint:** Drop the $\mathcal{O}(|w|^4)$ terms and separate real and imaginary parts in the expression in the exponent of (1).

5. Consider the vector field

$$\begin{aligned} x' &= x^2 y - x^5 \\ y' &= -y + x^2. \end{aligned}$$

Near $(0,0)$ there is a Center Manifold, the graph of $y = h(x) = ax^2 + bx^3 + \dots$. Write down the first order differential equation satisfied by the function h and find a and b.

6. Consider the product space $\mathbb{R}^n \times \mathbb{R}$ and let $P : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be projection onto the second factor, $P(x,t) = t$. For two t values, $t_1 < t_2$ define the "copies" of $\mathbb{R}^n$, $X = P^{-1}(t_1)$ and $Y = P^{-1}(t_2)$ and the mapping

$$T : X \to Y \quad \text{where} \quad y = T(x) = \phi(t_2, t_1, x)$$

and $\phi(t_2, t_1, x)$ is the solution $\phi(t, t_1, x)$ of $y' = f(t,y), \quad y(t_1) = x$, evaluated at t_2. Assume that $f \in C^1$ and the divergence,

$$\text{div } f \doteq \frac{\partial f_1}{\partial y_1} + \frac{\partial f_2}{\partial y_2} + \dots + \frac{\partial f_n}{\partial y_n} = 0.$$

Show (1) T is a diffeomorphism and (2) T is (Lebesgue)measure preserving.