COMPLEX ANALYSIS GRADUATE EXAM

FALL 2007

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose $a > 1$. Show that

$$\int_0^{2\pi} \frac{d\theta}{a + \sin\theta} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

2. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is entire. Show that the Taylor series of f at 0 converges to f uniformly on $\mathbb{C}$ if and only if f is a polynomial.

3. Let f be a one-to-one holomorphic function on the unit disc $B_1 = \{z \in \mathbb{C} : |z| < 1\}$ and let $D = f(B_1)$ be the image of B_1 under f. Similarly let D_r be the image under f of the open disc $B_r = \{z \in \mathbb{C} : |z| < r\}$ for $0 < r < 1$. Show that if $h : D \to D$ is a holomorphic mapping leaving the point $f(0)$ fixed then

$$h(D_r) \subseteq D_r \text{ for } 0 < r < 1.$$

4. Let $f(z)$ be continuous in $\mathrm{Re}\, z \geq 0$ and analytic in $\mathrm{Re}\, z > 0$. Let $g(x)$ be continuous in $\mathrm{Re}\, z \leq 0$ and analytic in $\mathrm{Re}\, z < 0$. Assume that $f = g$ on $\mathrm{Re}\, z = 0$. Prove that f and g are differentiable in the closure of their domains and that $\partial f/\partial x = \partial g/\partial x$ at the origin.