COMPLEX ANALYSIS GRADUATE EXAM
Fall 2008

Answer all five questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Map the region $\{|z| < 1\} \setminus \{|z - \frac{1}{2}| < \frac{1}{2}\}$ conformally to the upper half plane.

(2) Evaluate the integral

$$\int_0^\infty \frac{1}{x^{1/2}(1+x^2)}\, dx.$$

(3) Assume f is meromorphic in $\{|z| \leq 1\}$, and $|f(z)| = 1$ for all z with $|z| = 1$. Show that f is a rational function.

(4) A *fixed point* of a mapping f is a point z such that $f(z) = z$. Let $G = (0,1)^2$ be the open until square in $\mathbb{C}$. Show that if a holomorphic map $f : G \to G$ has two distinct fixed points, then it is the identity mapping.

(5) Let f be analytic in the unit disc, satisfying

$$M = \int_{\{|z|<1\}} |f(z)|\; dx\; dy < \infty.$$

Show that for all $|z| < 1$,

$$|f(z)| \leq \frac{M}{\pi(1-|z|)^2}.$$