MATH 505a GRADUATE EXAM

SPRING 2008

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) Let $Y \geq 0$ be a random variable with density f, and let X be another random variable. Assume X and Y have finite variances. Show that $E(XY) = \int_0^\infty E(XI_{[Y \geq t]})\, dt$. HINT: First express $E(XI_{[Y \geq t]})$ as an integral involving $E(X \mid Y = y)$.

(2) Let $\mathbf{X} = (X_1, .., X_m)$ and $\mathbf{Y} = (Y_1, .., Y_m)$ be random vectors with covariance matrices $\Sigma_{\mathbf{X}}$ and $\Sigma_{\mathbf{Y}}$.

(a) The *cross-covariance matrix* of $\mathbf{X}$ and $\mathbf{Y}$ is given by $C_{ij} = \mathrm{cov}(X_i, Y_j)$. For vectors $\mathbf{a}, \mathbf{b}$, express $\mathrm{var}(\mathbf{a} \cdot \mathbf{X} + \mathbf{b} \cdot \mathbf{Y})$ in terms of $\mathbf{a}, \mathbf{b}, \Sigma_{\mathbf{X}}, \Sigma_{\mathbf{Y}}$ and C.

(b) Suppose that for some vectors $\mathbf{a}, \mathbf{b}$ and some $k \in \mathbb{R}$, we have $\mathrm{var}((\mathbf{a}+u\mathbf{b}) \cdot \mathbf{X}) = ku$ for all $u \in \mathbb{R}$. Show that there are constants c_1, c_2 such that $P(\mathbf{a} \cdot \mathbf{X} = c_1) = P(\mathbf{b} \cdot \mathbf{X} = c_2) = 1$, and determine the value of k.

(3) Let U be a standard Cauchy random variable, that is, the density of U is $f_U(x) = \frac{1}{\pi} \frac{1}{1+x^2}, x \in \mathbb{R}$.

(a) Show that U and $1/U$ have the same distribution.

(b) Show that $E|U|^\alpha \geq 1$ for all $0 < \alpha < 1$. HINT: $1 = U \cdot \frac{1}{U}$.

(4) A sequence $X_1 X_2 \dots X_n$ is said to have a local maximum at 1 if $X_1 > X_2$, a local maximum at i (for $1 < i < n$) if both $X_i > X_{i-1}$ and $X_i > X_{i+1}$, and a local maximum at n if $X_n > X_{n-1}$. Let N be the number of local maxima.

(a) Find the mean and variance for N in each of the following cases:

(i) if $X_1 X_2 \dots X_n$ is a random permutation of the numbers $1, 2, \dots, n$, with all $n!$ permutations equally likely;

(ii) if $X_1, X_2, \dots, X_n$ are chosen independently and uniformly from the integers $\{1, 2, \dots, q\}$.

(b) Pick one of the two cases (i) or (ii) in (a), and show that $N/n \to 1/3$ in probability as $n \to \infty$.