MATH 505a GRADUATE EXAM

FALL 2007

Answer as many questions as you can. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. If you cannot do part (a) of a problem, you can still get credit for (b), (c) etc. by assuming the answer to (a). Start each problem on a fresh sheet of paper, and write on only one side of the paper.

(1) (Daniel Bernoulli, 1786) Of the $2n$ people in a collection of n couples, exactly m die, with all $\binom{2n}{m}$ possibilities equally likely. Find the expected number of surviving couples.

(2) Let X be a standard normal random variable, and let Y be independent of X with $P(Y = 1) = P(Y = -1) = 1/2$. Answer the following questions and *justify your answers*:
  (a) Is the random variable $Z = XY$ normally distributed?
  (b) Do X and Z have a nonzero correlation?
  (c) Does (X, Z) have a bivariate normal distribution?

(3) A sequence of mean-0 random variables $(X_n)_{n\in\mathbb{N}}$ is called *weakly stationary* if there is a function ϕ such that

$$E[X_iX_j] = \phi(|j-i|) < \infty \qquad \text{for all } i, j,$$

in other words this expected value only depends on the difference $|j - i|$. Suppose that for some such sequence, we have $\phi(k) \to 0$ as $k \to \infty$. Show that the weak law of large numbers is valid, that is, $\frac{X_1+\cdots X_n}{n}$ converges to 0 in probability.

(4) Consider a branching process with immigration: each generation is supplemented by an "immigrant" with probability p. This means that the size Z_n of the nth generation satisfies

$$Z_{n+1} = I_{n+1} + \sum_{i=1}^{Z_n} X_i,$$

where $I_{n+1} = 1$ with probability p, 0 with probability $1-p$, and the family sizes X_i are i.i.d. with generating function $G(s)$. We assume Z_n, I_{n+1} and $\{X_i\}$ are independent. Let $G_n(s)$ be the generating function of Z_n and let $\mu_n = EZ_n$.
  (a) Show that $G_{n+1}(s) = (ps + (1-p))G_n(G(s))$. HINT: Condition on Z_n.
  (b) Show that $\mu_{n+1} = p + \mu_n\mu$.
  (c) If $\{\mu_n\}$ converges to a finite limit μ_∞, then what is μ_∞, in terms of p and μ?