Algebra Qualifying Examination, Spring 2008

Directions

This exam consists of 7 problems. Please do 6 of them and show your work. If you are using a well-known result in your proof, please refer to it by name. Good Luck!

1. Let G be a finite group with A a normal subgroup of G. Let P be a Sylow p-subgroup of A. (a) If $g \in G$, show that $gPg^{-1} = xPx^{-1}$ for some $x \in A$. (b) Prove that $G = AN_G(P)$. (c) Prove that if $[G : A] = p$, then the number of Sylow p-subgroups of G that contain A is equal to $[N_G(P) : N_G(Q)]$ where Q is a Sylow p-subgroup of G containing P.

2. Let G be a finite group of order n. Let $f : G \to S_n$ be the regular representation of G. (a) Show that the image of f is contained in the alternating group if and only if the Sylow 2-subgroup of G is not cyclic. (b) Use (a) to show that if G has a cyclic nontrivial Sylow 2-subgroup, then G contains a normal subgroup N of odd order (hint: show first that G has a normal subgroup of index 2).

3. Let A be a (commutative) integral domain that is not a field and let K be the quotient field of A.

(a) Prove that K is not finitely generated as an A-module. Hint: use Nakayama's lemma.

(b) Can K ever be finitely generated as an A-algebra?

4. Let K be a field and p be a prime number that is not equal to the characteristic of K. Assume that K does not contain a primitive p-th root of unity. Let $a \in K^*$ that is not a p-th power and let b be an

element of an separable closure of K with $b^p = a$. Consider the field $L = K(\zeta, b)$, where ζ is a primitive p-th root of unity. Prove that L/K is a Galois extension and that $Gal(L/K)$ is isomorphic to the group of 2 by 2 matrices

$$\begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix}$$

where $r \in \mathbb{Z}/p\mathbb{Z}$ and $s \in (\mathbb{Z}/p\mathbb{Z})^*$. Is this group abelian?

5. Let $R = \mathbb{C}[x, y]$ and consider the two ideals $I = (2x + y)$ and $J = (x^2 - y)$. (a) show that I and J are both prime ideals of R, and that each of them is the intersection of all of the maximal ideals containing it.

(b) Consider the ideal $I + J$. Is it a prime ideal?

(c) Same question for $I \cap J$.

(hint: you can get a lot of intuition for this problem by thinking about the analogous varieties in $\mathbb{R}^2$).

6. (i) Let A be a finitely generated abelian group. If ℓ is a prime number, let $A[\ell]$ denote the set of elements of A that are killed by ℓ. Then $A[\ell]$ and $A/\ell A$ are finite groups, say of orders n_1 and n_2, respectively. Express the difference $n_2 - n_1$ in terms of other invariants of A.

(ii) If A is an abelian group such that the groups $A[\ell]$ and $A/\ell A$ are finite, is A necessarily finitely generated? Give a proof or a counterexample.

7. Let R be a (left) Artinian ring which is an algebra over the field k. Assume that every $r \in R$ is algebraic over k, with a minimal polynomial of the form

$$x^n + a_{n-1}x^{n-1} + ... + a_1x + a_0,$$

such that either a_0 or a_1 is non-zero. Show that R is a direct sum of division rings.