ALGEBRA QUALIFYING EXAM, Fall 2008

1. Let p, q be odd primes with $p > 7$ and $q > 8p$. Let G be a group of order $8pq$.

(a) Show that G has a normal subgroup of order pq.
(b) Show that G has a normal subgroup of index 2.
(c) Show that G has a nontrivial center.

2. Let $G = L_1 \times \ldots \times L_t$, for $t > 1$, where all of the L_i are simple groups.

(a) Assuming that all of the L_i are nonabelian, prove that the only normal subgroups of G are direct products of some subset of the L_i. (Hint: Let N be a normal subgroup of G and show that if the ith projection of N into L_i is nontrivial, then N contains L_i).

(b) Now suppose that all $L_i \cong L$, with L simple (possibly abelian). Show that there is no nontrivial proper subgroup of G which is invariant under all automorphisms of G. (Hint: Consider the abelian and nonabelian cases separately.)

(c) Suppose that $G = L \times L$ with L a nonabelian simple group. Let $D = \{(x, x) | x \in L\}$ be the diagonal subgroup. Show that D is a maximal subgroup of G.

3. Consider $f(x) = x^4 + x^2 + 9 \in \mathbb{Q}[x]$.

(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$. (Hint: first show that the only possible factors are quadratic, and then see what happens when x is replaced by $-x$.)

(b) Find the Galois group of $f(x)$ over $\mathbb{Q}$.

(c) Describe the splitting field of f over $\mathbb{Q}$ and the intermediate fields.

4. Let R be a commutative Noetherian ring. Show that any surjective ring endomorphism $\phi\colon R \to R$ is an automorphism.
(Hint: consider the iterations $\phi, \phi^2, \phi^3, \ldots$.)

5. Let I be the ideal

$$I = (x^{37}y^{31}z^{29}t^{23}, x^3 + y^5, y^7 + z^{11}, z^{13} + t^{17}) \subset \mathbb{C}[x, y, z, t].$$

If $f(x, y, z, t)$ is any polynomial without constant term show that some power of f is in I.

6. Let A be a finite-dimensional algebra over $\mathbb{C}$. Show that if $x, y \in A$ such that $xy = 1$, then also $yx = 1$.

7. Let A, B, C be finitely generated modules over a PID R. Show that B is isomorphic to C if and only if $A \oplus B$ is isomorphic to $A \oplus C$.