# ALGEBRA EXAM FALL 2007

1. Let $G$ be a group of order 105.
   (a) Show $G$ has a normal subgroup of index 3.
   (b) Show $Z(G) \neq 1$.
   (c) Determine all possibilities for $G$.

2. Let $p$ be a prime. A group $G$ is called $p$-divisible if the map $x \to x^p$ is surjective. Suppose that $G$ is a finitely generated abelian group. Show that $G$ is $p$-divisible if and only if $G$ is finite and $p$ does not divide the order of $G$.

3. Let $R = \mathbb{C}[x_1, \ldots, x_n]$. Suppose that $f \in R$ is irreducible. If $g(a) = h(a)$ whenever $f(a) = 0$, show that $g + (f) = h + (f)$ in $R/(f)$.

4. Let $F$ be a field. Suppose that $A$ is an $F$-subalgebra of $M_n(F)$ containing the identity of $M_n(F)$.
   (a) If $A$ is a domain, show that $A$ is a division algebra and $\dim A \leq n$.
   (b) If $A$ is simple, show that $(\dim A) | n^2$ (hint: Let $V$ be the space of column vectors of size $n$ over $F$ – this is a left $M_n(F)$-module of dimension $n$; show that $V$ is a direct sum of say $s$ isomorphic copies of a simple $A$-module $U$. Relate the dimension of $A$ and the dimension of $U$).

5. Let $p$ be a prime. Let $F := \mathbb{F}_{p^n}$ be the field of size of $p^n$. Let $f(x) \in F[x]$ be irreducible of degree $t$.
   (a) Show that the splitting field for $f$ has size $p^{nt}$.
   (b) If $n = 1$, show that $f(x) | (x^{p^m} - x)$ if and only if $t | m$.
   (c) How many irreducible polynomials of degree 6 are there over $\mathbb{F}_2$?

6. Let $R$ be a commutative ring with 1. Assume that $R = a_1 R + \ldots + a_n R$ for some $a_i \in R$. Let $M =: \{(r_1, \ldots, r_n) \in R^n | \sum a_i r_i = 0\}$. Show that $M$ is a projective $R$-module and can be generated by $n$ elements as an $R$-module.