Fall 2006 Math 541a Exam

1. Let $X_1, \cdots, X_n$ be independent identically distributed samples taking values 0 or 1 with distribution

$$P(X_i = 1) = 1 - P(X_i = 0) = p.$$

Suppose that

$$T_n = \sum_{i=1}^{n} I(X_i = 1, X_{i+1} = 1)$$

is observed.

(a) Calculate the expectation T_n.

(b) Find the method of moment estimator $\hat{p_n}$ for p based on T_n.

(c) Calculate the variance of T_n.

(d) Find the (properly scaled and centered) non-trival asymptotic distribution of $\hat{p_n}$.

2. Suppose $X_1, \cdots, X_n$ are i.i.d. samples with density $f(x, \theta)$, where $\theta \in E$, an open set in $\mathbf{R}$. Suppose $E_\theta(X_1) = \theta$. Let $\text{Var}_\theta(X_1) = \sigma^2(\theta) < \infty$, be integrable in θ. A transformation h: $E \to R$ such that $h' > 0$ is called variance stabilizing if for all θ, $\sqrt{n}(h(\bar{X}) - h(\theta)) \to N(0, b^2)$, where b is a constant.

(a) Show that h is variance stabilizing if and only if $h'(\theta) = b\sigma^{-1}(\theta)$ for some b and all $\theta \in E$.

(b) Let $X_1, \cdots, X_n$ be the indicators of n binomial trials with probability of success rate θ. Show that the only variance stabilizing transformation h such that $h(0) = 0$, $h(1) = 1$, and $h'(t) \geq 0$ for all t, is given by $h(t) = (2/\pi)\sin^{-1}(\sqrt{t})$.

(c) (Continue) Let $S = \sum_{i=1}^{n} X_i$ and $\bar{X} = S/n$. Show that $\sin^{-1}(\sqrt{\bar{X}}) \pm z(1-0.5\,\alpha)/(2\sqrt{n})$ is an approximate level $(1-\alpha)$ confidence interval for $\sin^{-1}(\sqrt{\theta})$, where $z(\alpha)$ is the α-th quantile of the standard normal distribution.