PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Spring 2022

Partial credit will be awarded, but in the event that you cannot fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $u \in C^2(U)$ be a subharmonic function (i.e. $\Delta u \geq 0$ in U).

   (a) Show that
   $$u(x) \leq \fint_{\partial B(x,r)} u(y)dS(y)$$
   for every $x \in U$ and $r > 0$ such that $\overline{B(x,r)} \subset U$.

   (b) Show that if U is open with a C^1 boundary and $u \in C(\overline{U})$ then the maximum principle
   $$\max_{\overline{U}} u = \max_{\partial U} u$$
   holds.

   (c) Show that if additionally $v \in C^2(U) \cap C(\overline{U})$ is superharmonic (i.e. $\Delta v \leq 0$ in U) and $u \leq v$ on ∂U then $u \leq v$ in U.

2. Find all classical solutions of the equation
   $$u_t - \Delta u = t \sin 2x \, \sin 2y$$
   in $U = (0, \pi)^2 \times \mathbb{R}_+$, such that
   $$\begin{cases} u|_{\partial U} = 11 \\ u(x, y, 0) = \sin x \, \sin 2y - 3 \sin 3x \, \sin y + 11 & \text{for } (x, y) \in U. \end{cases}$$

3. Let $u(x,t)$ be a solution of the wave equation $u_{tt} = u_{xx}$ for $(x,t) \in \mathbb{R} \times (0, \infty)$ such that $u(x, 0) = g(x)$, $u_t(x, 0) = 0$, where $g \in C^2(\mathbb{R})$ has compact support. Show that
   $$v(x,t) := \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{s^2}{4t}} u(x, s) ds$$
   solves the heat equation with initial condition $v(x, 0) = g(x)$.