1. Let $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ where $\mu \in \mathbb{R},\ \sigma^2 > 0$, with both parameters unknown.
   (a) Find the complete sufficient statistics for (μ, σ^2), and the uniform minimum variance unbiased estimator $s_{0,n}^2$ of σ^2.
   (b) Find the maximum likelihood estimator s_n^2 of σ^2. Is it biased?
   (c) Which of the two estimators $s_{0,n}^2,\ s_n^2$ has smaller mean squared error? Recall that the MSE of an estimator $\widehat{\theta}$ of parameter θ is defined as $\text{MSE}(\widehat{\theta}) = \mathbb{E}(\widehat{\theta} - \theta)^2$. Based on your answer, explain which estimator you would prefer and why.
   (hint: you may use the fact that $\sum_{j=1}^{n}(X_j - \bar{X}_n)^2$ has χ^2 distribution with $n-1$ degrees of freedom, and that the variance of a χ^2 random variable is twice the number of degrees of freedom)

2. Let $X_1, X_2, \ldots$ be i.i.d random variables, each with density given by f_θ, where $\theta \in \Theta \subseteq \mathbb{R}$ is an unknown parameter. For any $n \geq 1$, a maximum likelihood estimator Y_n satisfies

$$\prod_{i=1}^{n} f_{Y_n}(X_i) = \sup_{\theta \in \mathbb{R}} \prod_{i=1}^{n} f_\theta(X_i).$$

   (a) State (without the proof) the most general version of the maximum likelihood estimator (MLE) consistency theorem familiar to you which concludes that $Y_1, Y_2, \ldots$ converges in probability to the true value of θ, under some assumptions.
   (b) Consider the density of the uniform distribution on $[0, 1]$, that is $f(x) = 1$ for all $x \in [0, 1]$ and $f(x) = 0$ otherwise, and define $f_\theta(x) = f(x - \theta)$, $\theta \in \mathbb{R}$. Find an MLE Y_n of θ in this case. Is it unique?
   (c) Is the MLE that you found consistent? If not, explain why. If it is consistent, can it be proven using the result that you stated in part (a)? If not, explain why and establish consistency in a different way.