Geometry and Topology Graduate Exam
Spring 2022

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let B^3 be the solid ball of radius 2 centered at the origin in $\mathbb{R}^3$. Let S^1 be the unit circle in the xy-plane. Compute the homology groups of $B^3 - S^1$.

Problem 2. Prove that the fundamental group and the homology groups (in every degree) of $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are isomorphic, but that the homology groups of their universal covering spaces are not.

Problem 3. Consider the vector fields $v_k = x^k \frac{\partial}{\partial x}$ on $\mathbb{R}$, where $k \geq 0$.

(1) Find the Lie bracket $[v_i, v_j]$.
(2) Do the flows of v_i and v_j commute?
(3) Find the flow of v_2.
(4) Is v_2 complete on $\mathbb{R}$, i.e. does every one of its flow curves exist for all time?

Problem 4. (*intersecting with a plane and a cylinder*) Let M^m be any submanifold of $\mathbb{R}^N$ with $N \geq 3$. Show that there exist real numbers $a, b > 0$ so that the intersection $M \cap \{x_N = a\} \cap \{x_1^2 + x_2^2 = b^2\}$ is a submanifold (of M and hence $\mathbb{R}^N$) of dimension $m - 2$. **Note**: The condition $N \geq 3$ is not strictly necessary, it is imposed simply to ensure that x_N, x_1, and x_2 are all different coordinates to avoid the case $\{x_N = a\} \cap \{x_1^2 + x_2^2 = b^2\}$ is empty (in which case the result would be vacuously still true).

Problem 5.

Say whether each assertion is true or false, with complete justification.

(1) $S^3 \setminus \{3 \text{ points}\}$ is homotopy equivalent to $S^2 \setminus \{2 \text{ points}\}$.
(2) Any continuous map $S^2 \to S^1 \times S^1$ is null homotopic.

Problem 6. Let $f : M^n \to N^n$ be a smooth map between compact oriented manifolds of the same dimension $n > 1$, and suppose f factors as $M^n \to S^1 \times \mathbb{R}^{n-1} \to N^n$ (with $n - 1 > 0$). Show that in any neighborhood U of any point $x \in N^n$, there exists a point $y \in U$ with $f^{-1}(y)$ having an even number of points.

Problem 7. Let M^m be a manifold and $\lambda \in \Omega^1(M)$ a closed 1-form ($d\lambda = 0$). Suppose $f : S^2 \to M$ is a smooth map, where S^2 is the standard unit 2-sphere in $\mathbb{R}^3$, and let $f|_{S^1}$ denote the restriction of f to the equator circle $S^1 = \{z = 0\} \cap S^2 \subset S^2$. Show that

$$\int_{S^1} (f|_{S^1})^* \lambda = 0.$$