REAL ANALYSIS GRADUATE EXAM

Spring 2022

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Prove that for almost all $x \in [0,1]$, with respect to the Lebesgue measure, there are at most finitely many rational numbers in a reduced form $\frac{p}{q}$, where $q \geq 2$, so that

$$\left| x - \frac{p}{q} \right| < \frac{1}{(q \log q)^2}.$$

(Hint: Consider intervals of lengths $1/(q \log q)^2$ centered at rational points p/q.)

2. Let $S \subseteq \mathbb{R}$ be closed, and let $f \in L^1([0,1], m)$, where m denotes the Lebesgue measure on $[0,1]$. Assume that for all measurable $E \subseteq [0,1]$ with $m(E) > 0$ we have

$$\frac{1}{m(E)} \int_E f(x)\, dx \in S.$$

Prove that $f(x) \in S$ for a.e. $x \in [0,1]$.

3. Evaluate the limit

$$\lim_n \int_0^1 \frac{1 + nx}{(1+x)^n}\, dx.$$

4. Assume that (X, μ) is a finite measure space and $\{f_n\}_{n=1}^{\infty}$ a sequence of nonnegative measurable functions on X. Prove that $f_n \to 0$ in measure if and only if

$$\lim_n \int_X \frac{f_n(x)(2 + f_n(x))}{(1 + f_n(x))^2}\, d\mu(x) = 0.$$