COMPLEX ANALYSIS GRADUATE EXAM

Spring 2022

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Use residues to compute

$$\int_0^\infty \frac{dx}{1+x^{2022}}.$$

2. Map $\Omega = \{z \in \mathbb{C} : \mathbb{R}\mathrm{e}\, z > 0,\ \mathbb{I}\mathrm{m}\, z < 1\}$ conformally to $\mathbb{D}\backslash[0,1]$ (or the reverse), where $\mathbb{D}$ denotes the unit disk centered at 0.

3. Let $n \in \mathbb{N}$ and

$$f(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + 15z^8, \qquad z \in \mathbb{D},$$

where $\mathbb{D}$ is the unit disk centered at 0. Determine the number of zeros, with counted multiplicities, of f in $\mathbb{D}$.

4. Show that if f is entire and nowhere zero, then there exists an entire function g such that $f(z) = g(z)^2$ for all $z \in \mathbb{C}$.