ALGEBRA QUALIFYING EXAM SPRING 2022

PROBLEM 1. Consider the polynomial ring $\mathbb{C}[x_{ij}, 1 \le i, j \le n]$ as the algebra of polynomial functions on the space of $n \times n$ matrices $M_n(\mathbb{C})$. Let $\mathcal{N} \subset M_n(\mathbb{C})$ be the set of all nilpotent matrices. Introduce n polynomials $P_j \in \mathbb{C}[x_{ij}, 1 \le i, j \le n]$, $1 \le j \le n$, defined by $P_j(A) = \mathrm{Tr} A^j$. Prove that a polynomial $Q \in \mathbb{C}[x_{ij}, 1 \le i, j \le n]$ vanishes on $\mathcal{N}$ if and only if some power of Q belongs to the ideal $(P_1, P_2, ..., P_n)$.

PROBLEM 2. Let $R \subset S$ be an integral ring extension. Prove that $a \in R$ is invertible as an element of R if and only if it is as an element of S.

PROBLEM 3. Let R be a commutative ring and M a finitely generated R-module.

(a) Prove that if R is a principal ideal domain, then M is projective if and only if M is torsion free.

(b) Answer the question: does the assertion (a) remain valid if R is assumed to be a local domain?

PROBLEM 4. Show that the center of a simple ring is a field and that the center of a semi-simple ring is a finite direct product of fields.

PROBLEM 5. Set $n = |\mathrm{SL}_2(\mathbb{F}_7)|$. For each $p|n$, find a Sylow p-subgroup of $\mathrm{SL}_2(\mathbb{F}_7)$.

PROBLEM 6. Find the Galois group of the polynomial $x^4 - 4x^2 - 21$ (over $\mathbb{Q}$). Answer the question: is this polynomial solvable in radicals?