Math 507a Qualifying Exam, Spring 2022

Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

Problem 1 : Let $E_1, E_2, \cdots$ be arbitrary events. Let $G := \limsup_n E_n$. Show that $\mathbb{P}(G) = 1$ if and only if $\sum_n \mathbb{P}(A \cap E_n) = \infty$ for all events A having $\mathbb{P}(A) > 0$.

Problem 2: Let $X_0, X_1, \cdots$ be i.i.d. and continuous. Let $N = \inf\{n \geq 1 : X_n > X_0\}$. Show that

$$\mathbb{P}(N > n) = \frac{1}{n+1},$$

and find $\mathbb{E}\, N$.

Problem 3: Let $X_1, \cdots$ be independent random variables such that $\sum_n n^{-2} \operatorname{Var}(\mathrm{X_n}) < \infty$. Prove that there is a random variable Y so that

$$\sum_{k=1}^{n} \frac{X_k - \mathbb{E}\, X_k}{k} \to Y$$

with probability 1. Do NOT assume that the X_i have the same distribution. Hint: Consider an appropriate S_n and use the Kolmogorov inequality for

$$\mathbb{P}(\sup_{k \geq n} |S_k - S_n| > \epsilon).$$