Math 505a Qual, Spring 2022

Instructions: Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. If you find that a calculation leads to something impossible, such as a negative probability or variance, indicate that something is wrong, but show your work anyway. Be aware of the passage of time, so that you can attempt all three problems. When a problem asks you to find something, you are expected to simplify the answer as much as possible.

Problem 1:

a) Let X_1, X_2, X_3 be independent exponential random variables with parameter $\lambda = 1$. So $P(X_i > x) = e^{-x}$, $x > 0$. Find

$$E\left(\frac{X_1}{X_1 + X_2 + X_3}\right).$$

b) Let (X, Y) be independent uniforms on $[0, 1]$. Find the joint density function of X and $V = X + Y$. Find $f(x|v)$, the density function of X conditional on $V = v$. Also, find $E(X|V)$.

Problem 2: In an election, candidate A receives n votes, and candidate B receives m votes, where $n > m$. Assuming that all $\binom{m+n}{m}$ orderings are equally likely, show that the probability that A is always ahead in the count of votes is $(n - m)/(n + m)$.

Problem 3: Let n be a positive integer with prime factorization $n = p_1^{m_1} \cdots p_k^{m_k}$ for distinct primes $p_1, \cdots, p_k$ with $m_1, \cdots, m_k > 0$. Choose an integer N uniformly at random from the set $\{1, 2, \cdots, n\}$. Show that the probability that N shares no common prime factor with n is equal to

$$(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \cdots (1 - \frac{1}{p_k}).$$

(Hint: use inclusion-exclusion)