ODE EXAM - Spring 2021

The exam has **four** problems. Each problem is worth 10 points. Do all four problems.

1. Show that, for every $a \in [-1, 1]$, the solution $y = y(t)$ of the equation

$$y' = e^y - 1 - 2y,$$

with initial condition $y(0) = a$, is defined for all $t > 0$ and satisfies $|y(t)| \leq |a|$.

2. Consider the linear system

$$X'(t) = \big(A + \varepsilon B(t)\big)X(t),$$

where the matrix $A \in \mathbb{R}^{n\times n}$ is constant and symmetric, ε is a real number, and the entries of the matrix $B = B(t) \in \mathbb{R}^{n\times n}$ are bounded continuous functions of t for all $t \geq 0$. Denote by $|X(t)|$ the Euclidean norm of the vector $X(t)$.

Show that if all eigenvalues of A are strictly negative and $|\varepsilon|$ is sufficiently small, then all solutions of the system satisfy

$$\lim_{t\to+\infty} |X(t)| = 0.$$

3. For the nonlinear system

$$\begin{cases} x' = -\mu x + xy^2 + x^3 \\ y' = -\mu y + x^2 y + y^3, \end{cases}$$

determine the type of bifurcation as μ increases from -1 to 1.

4. Let $f = f(t)$ be a continuous periodic function with period 1. Consider the second-order equation

$$y''(t) + f(t)y(t) = 0,$$

and let $u = u(t)$ and $v = v(t)$ be two solutions of the equation with initial conditions $u(0) = v'(0) = 1$, $u'(0) = v(0) = 0$. Show that if $u(1) + v'(1) = 2$, then the equation has a periodic solution with period 1.