PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM
Spring 2021

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Solve the following initial value problems and verify your solutions

   (a) $2u_x + 3u_t = u^2, \quad u(x,0) = h(x),\ t > 0, \quad x \in \mathbb{R}, \quad$ (here h is given)

   (b) $u_t = x^2 u u_x, \quad u(x,0) = x,\ t > 0, \quad x \in \mathbb{R}$

   (c) $xu_x + yu_y + u_z = u, \quad u(x,y,0) = h(x,y), \quad z > 0, \quad (x,y) \in \mathbb{R}^2 \quad$ (here h is given)

   (d) $u_x^2 + u_y^2 = u^2 \quad$ (here find the characteristic equations only).

2. Let B be the unit disc in $\mathbb{R}^2$, and ∂B the unit circle. Let f and g be two *analytic* functions defined on ∂B.

   (a) Prove that for any point $x \in \partial B$, there exists a neighborhood U of x and a function u harmonic in $U \cap B$, such that $u = f$, and the outward normal derivative $\partial_\nu u = g$, on $U \cap \partial B$.

   (b) Does there always exist a function u harmonic in B, such that $u = f$ and $\partial_\nu u = g$ on ∂B? Why or why not?

3. Let $\theta(x,t)$ be a strictly positive smooth solution of the following heat equation

$$\theta_t - \nu\theta_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0$$

   where $\nu > 0$ is a positive constant.

   (a) Show that $u = -\frac{2\nu\theta_x}{\theta}$ satisfies

$$u_t + uu_x - \nu u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0. \tag{1}$$

   (b) For $u_0 \in C_c^2(\mathbb{R})$, find a solution to (1) with initial data $u(x,0) = u_0$ for which $\lim_{t\to\infty} u(x,t) = 0$.