1. (Wilk's Theorem) Let $X_1, \ldots, X_n$ be independent random variables distributed as $\mathcal{P}(\lambda), \lambda \in \Lambda = (0, \infty)$, that is, with probability mass function

$$P_\lambda(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, \ldots$$

For $\lambda_0 \in \Lambda$, we wish to test $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$. It may be useful to recall that the mean and variance of the $\mathcal{P}(\lambda)$ distribution are both equal to λ.

1.1. Write out the Generalized likelihood ratio test statistic G_n for this instance.

1.2. Verify directly that the conclusion of Wilk's theorem holds, that is, asymptotically as $n \to \infty$ that $-2 \log G_n$ has a χ^2 distribution, and specify its degrees of freedom. You may use the fact that $\overline{X}/\lambda_0 = 1 + (\overline{X} - \lambda_0)/\lambda_0$, and also apply $x - x^2/2$ as an approximation to $\log(1 + x)$ for x small, without further justification.

1.3. Using direct methods as in the previous part, develop an asymptotic for the power of the test in terms of the non-central χ^2 statistic for the sequence of alternatives of the form $\lambda_n = \lambda_0 + \delta/\sqrt{n}$; make sure you specify both the degrees of freedom and the non-centrality parameter. (Recall that the sum $(Z_1 + \mu_1)^2 + \cdots + (Z_k + \mu_k)^2$, where $Z_1, \ldots, Z_k$ are iid $\mathcal{N}(0, 1)$, has a non central χ^2 distribution on k degrees of freedom with non-centrality parameter $\mu_1^2 + \cdots + \mu_k^2$.)

2. (Jackknife) Let $X_1, \ldots, X_n$ be independent with common distribution function depending on $\theta \in \Theta \subset \mathbb{R}$, unknown.

2.1. Let $W_1, W_2, \ldots$ be a sequence of estimators for a parameter $\theta \in \Theta$ so that for any $n \geq 1$, $W_n = t_n(X_1, \ldots, X_n)$ for some $t_n : \mathbb{R}^n \to \Theta$. For any $n \geq 1$, define the **jackknife estimator** of θ to be

$$Z_n := nW_n - \frac{n-1}{n} \sum_{i=1}^{n} t_{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n).$$

Assume that $W_1, W_2, \ldots$ are asymptotically unbiased in the sense that there exists $a, b \in \mathbb{R}$ such that

$$EW_n = \theta + a/n + b/n^2 + O(1/n^3), \qquad \forall\, n \geq 1. \qquad (*)$$

Show that if $b = 0$ and the $O(1/n^3)$ term is zero in $(*)$, then Z_n is unbiased for θ.

2.2. Show, generally that when $(*)$ holds,

$$EZ_n = \theta + O(1/n^2), \quad \forall\, n \geq 1.$$

2.3. Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli random variables with parameter $0 < \theta < 1$. The MLE for θ is the sample mean, so by the Functional Equivariance Property of the MLE, the MLE for θ^2 is

$$W_n := \Big(\frac{1}{n} \sum_{i=1}^{n} X_i\Big)^2.$$

(You can take this fact as given.) Show that W_n is a biased estimate of θ^2, but that the jackknife estimator of θ^2 is unbiased.