1. Let $X_1, \dots, X_n$ be i.i.d. observations with normal distribution $N(\mu, 1)$, where it is known that $\mu \geq 0$.

   1.1. Find the maximum likelihood estimator $\widehat{\mu}$ of μ.

   1.2. Derive the asymptotic distribution, as $n \to \infty$ of $\sqrt{n}(\widehat{\mu} - \mu)$, when the true mean satisfies $\mu > 0$; please provide a direct proof not relying on any general result about MLEs.

   1.3. Derive the asymptotic distribution, as $n \to \infty$, of $\sqrt{n}(\widehat{\mu} - \mu)$, that is, the limit of the probability $P(\sqrt{n}(\widehat{\mu} - \mu) \leq x)$ for all values of $x \in \mathbb{R}$, when the true mean μ is equal to 0.

   1.4. Why would typical general results about the asymptotic distribution of maximum likelihood estimators not apply in part (3)?

2. 2.1. Show that if W is UMVU for a parameter $\theta \in \mathbb{R}$, then for any U that is an unbiased estimator of 0 with finite variance, that is, $E_\theta[U] = 0$ and $E_\theta[U^2] < \infty$ for all $\theta \in \mathbb{R}$, we have $E_\theta[WU] = 0$ for all $\theta \in \mathbb{R}$. (Hint: Construct a family of unbiased estimators of θ using W and U.)

   2.2. Let X be a sample from the uniform distribution on the interval $[\theta - 1/2, \theta + 1/2]$, where $\theta \in \mathbb{R}$ is unknown. Find a continuous function $u(\cdot)$ such that $U = u(X)$ is an unbiased estimator of zero and $P_\theta(U \neq 0) = 1$. (Hint: consider periodic functions.)

   2.3. Let $g : \mathbb{R} \to \mathbb{R}$ be a nonconstant differentiable function of $\theta \in \mathbb{R}$. Show that there does not exist a UMVU of $g(\theta)$ of the form $w(X)$, a function of the observation X that is uniformly distributed over the interval $[\theta - 1/2, \theta + 1/2]$. You may assume you are given an unbiased estimator U of zero that satisfies the conditions in part (2), and that $W = w(X)$ with $w(x)$ continuous in x. (Hint: Use part (1) and differentiate the equalities $E_\theta[U] = 0$ and $E_\theta[WU] = 0$ with respect to $\theta \in \mathbb{R}$.)