Geometry and Topology Graduate Exam

Spring 2021

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Let $X = S^1 \times S^1 - \{p, q\}$, with $p \neq q$, be the twice punctured 2–dimensional torus.

(1) Compute the homology groups $H_n(X, \mathbb{Z})$.
(2) Compute the fundamental group of X.

Problem 2. Let X be the figure eight, union of two circles meeting in exactly one point x_0. Recall that the fundamental group $\pi_1(X; x_0)$ is the free group on two generators a and b, respectively going once around the first and the second circle. Draw a covering $p\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is connected and $p_*(\pi_1(\widetilde{X}; \widetilde{x}_0))$ is the subgroup $G \subset \pi_1(X; x_0)$ generated by the subset $\{a^2, b^2, aba, bab\}$.

Use this construction to decide whether this subgroup G is normal or not.

Problem 3. Let M be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u\colon T_xM \to \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

Problem 4.

Show that, if a map $f : S^n \to S^n$ has no fixed points, then its degree is equal to $(-1)^{n+1}$. Possible hint: Show that f is homotopic to a simple map.

Problem 5. Let T be the torus in $\mathbb{R}^3$ obtained by revolving the circle

$$\{(x, y, z) \in \mathbb{R}^3; (x-2)^2 + y^2 = 1 \text{ and } z = 0\}$$

around the y-axis. Compute the integral

$$\int_T xdy \wedge dz - ydx \wedge dz + zdx \wedge dy.$$

Problem 6. Let $f : M \to N$ be a differentiable map between two connected compact orientable manifolds of the same dimension n. Suppose that there exists a nonempty open subset U such that $f^{-1}(U)$ can be written as a disjoint union $U_1 \coprod U_2 \coprod U_3$ for which each restriction $f|_{U_i} : U_i \to U$ is a diffeomorphism. Show that f is necessarily surjective.

Problem 7. Consider the differential 2–form $\omega = \frac{dx \wedge dy}{x^2+y^2}$ on $X = \mathbb{R}^2 - \{0\}$, and denote by $Y = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$ the unit circle inside X. Prove that, for the unit disk D^2 and for any smooth map $f\colon D^2 \to X$ which sends the boundary of the disc to Y,

$$\int_{D^2} f^*(\omega) = 0,$$

where $f^*(\omega) \in \Omega^2(D^2)$ (also denoted as $\Omega^2(f)(\omega)$) is the pull back of ω under f.