REAL ANALYSIS GRADUATE EXAM

Spring 2021

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Suppose that f is a bounded non-negative function on a measure space (X, μ) with $\mu(X) = \infty$. Prove that f is integrable if and only if

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \mu(\{x \in X : f(x) > 2^{-n}\}) < \infty.$$

2. Let f be a real valued function on $[0, 1]$. Prove that the set of points where f is continuous is Lebesgue measurable.

3. Let f be a Lebesgue integrable function on $\mathbb{R}$ and let $\beta \in (0, 1)$. Prove that

$$\int_0^{\infty} \frac{|f(x)|}{|x - a|^{\beta}} \, dx < \infty.$$

for a.e. $a \in \mathbb{R}$.

4. Let f_n be a sequence of real-valued function on $[a, b]$ so that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in [a, b]$. Let $V_a^b(f)$ be the total variation of f on $[a, b]$. Show that

$$V_a^b(f) \leq \liminf_{n\to\infty} V_a^b(f_n).$$