COMPLEX ANALYSIS GRADUATE EXAM

Spring 2021

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Evaluate

$$\int_{|z|=2} \frac{4z^7 - 1}{z^8 - 2z + 1} dz,$$

carefully justifying all the steps.

2. Let $\Omega \subseteq \mathbb{C}$ be an open set and $z_0 \in \Omega$. Suppose that f is holomorphic in $\Omega \backslash \{z_0\}$ and that z_0 is either a pole of order $m \in \mathbb{N}$ or a removable singularity for f whose removal results in z_0 being a zero of order m. Prove that z_0 is a first order pole of f'/f having residue either $-m$ or m.

3. Find a bijective analytic function which maps

$$\Omega = \left\{ z = x + iy \in \mathbb{C} : |z| < 1, \;\; y > -\frac{1}{\sqrt{2}} \right\}$$

to the unit disk $\mathbb{D} = \{z = x + iy \in \mathbb{C} : |z| < 1\}$.

4. Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Let $f \colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function with two unequal fixed points (i.e., $f(a) = a$ and $f(b) = b$ with $a \neq b$). Prove that $f(z) = z$ for $z \in \mathbb{D}$.