MATH 507a **QUALIFYING EXAM** **January 2021**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve.

(1) For μ, ν probability measures on Borel sets in $\mathbb{R}$, define

$$\rho(\mu, \nu) = \inf\{\epsilon > 0 : \mu(A) \leq \nu(A^\epsilon) + \epsilon \text{ for all closed } A \subset \mathbb{R}\}.$$

Here $A^\epsilon = \{x \in \mathbb{R} : d(x, A) < \epsilon\}$, with d being Euclidean distance. Show that if μ_n, μ are probabillity measures satisfying $\rho(\mu_n, \mu) \to 0$ then $\mu_n \to \mu$ weakly.

(2) Let $X, X_1, X_2, \ldots$ be independent and identically distributed with characteristic function φ, and let $S_n = X_1 + \cdots + X_n$.

(a) Suppose $\varphi'(0) = 0$. Show that $S_n/n \to 0$ in probability. HINT: What other modes of convergence are sufficient to establish this?

(b) Suppose X has the symmetric density

$$f(x) = c\frac{1}{x^2 \log|x|}, \quad |x| \geq 4,$$

where c is the appropriate normalizing constant. Using part (a), show: $E(X)$ does not exist, but $S_n/n \to 0$ in probability. HINT: In representing $\varphi(t)$, integrate separately over $|x| \in [4, \epsilon/t]$ and $|x| \in (\epsilon/t, \infty)$, for some small ϵ. You should only need to bound the resulting integrals, not calculate them explicitly.

(3) Let $\beta \in (0, 1)$ and let $X_1, X_2, \ldots$ be independent, with $X_i \sim \text{exponential}(i^{-\beta})$, that is, X_i has density $f(x) = i^{-\beta}e^{-i^{-\beta}x}$, $x \geq 0$. Let $M_n = \min_{i \leq n} X_i$. Show that

$$\sum_{n=1}^{\infty} M_n^\alpha$$

converges a.s. for all $\alpha > 1/(1 - \beta)$. HINT: Consider events $\{M_n \geq n^{-\gamma}\}$ for general $\gamma > 0$.