MATH 505a **QUALIFYING EXAM** **January 2021**

Answer all 3 questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all four problems.

(1) A permutation π on n symbols is said to have i as a fixed point if $\pi(i) = i$.

a) Find the probability p_n that a random permutation of n symbols has no fixed points. HINT: Principle of inclusion and exclusion. (Your answer may involve a finite sum, which you don't need to simplify.)

b) Let S be a subset of $\{1, 2, ..., n\}$ of size k. Find the probability that the set of fixed points of a random permutation on n symbols is equal to S, and find the probability that a permutation has exactly k fixed points. HINT: If you didn't find the values p_j in part (a), you can still give answers for (b) expressed in terms of one or more p_j's.

c) Show that as n tends to infinity, the distribution of the number of fixed points converges to a Poisson(1) distribution.

(2) Let $\{S_n, n \geq 0\}$ be symmetric simple random walk, that is, $S_n = \sum_{i=1}^n \xi_i$ with $\xi_1, \xi_2, \ldots$ i.i.d. satisfying $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. Let $T = \min\{n : S_n = 0\}$, and write P_a for probabilities when the walk starts at $S_0 = a$. By *basic probabilities for* $\{S_n\}$ we mean probabilities of the form $P_0(S_n = k), P_0(S_n \geq k)$, or $P_0(S_n \leq k)$, all of which corresponding to starting at $S_0 = 0$..

(a) For $a \geq 1, i \geq 1, n \geq 1$, express $P_a(S_n = i, T \leq n)$ and $P_a(S_n = i, T > n)$ in terms of finitely many basic probabilities. HINT: Reflection principle.

(b) For $a \geq 1, i \geq 1, n \geq 1$, show that

$$P_a(T > n) = \sum_{j=1-a}^{a} P_0(S_n = j).$$

HINT: Use (a) and look for cancellation.

(c) You may take as given that $P_0(S_{2m} = 2j) \sim 1/\sqrt{\pi m}$ as $m \to \infty$ for each fixed $j \in \mathbb{Z}$; here $\sim$ means the ratio converges to 1. Use this to find c, α such that $P_a(T > n) \sim c/n^\alpha$ as $n \to \infty$, where $a > 0$. Does c or α depend on a? HINT: It's enough to consider even n—why?

(3) Let X, Y be independent standard normal $N(0, 1)$ random variables.

(a) Find a for which $U = X + 2Y$ and $V = aX + Y$ are independent.

(b) Find $E(XY \mid X + 2Y = a)$ for all $a \in \mathbb{R}$. HINT: Use (a).