Numerical Analysis Preliminary Examination
Spring 2021

January 22, 2021

Problem 1. Consider the normed linear space $\mathbb{C}^n$ with vector norm $\| \cdot \|_2$ and let $A \in \mathbb{C}^{n\times n}$. The matrix norm $\| \cdot \|$ induced by the vector norm $\| \cdot \|_2$ is defined as $\| A \| = \sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.

(a) Show that $\| A \| = \sup_{\|x\|_2=1} \| Ax \|_2$.

(b) Show that in fact for each $A \in \mathbb{C}^{n\times n}$ there exists a $y \in \mathbb{C}^n$ with $\| y \|_2 = 1$ such that $\| A \| = \| Ay \|_2$.

(c) Use part (b) above and the eigenvalues $\{\lambda_j\}_{j=1}^n$ and corresponding eigenvectors $\{u_j\}_{j=1}^n$ of the matrix $A^H A \in \mathbb{C}^{n\times n}$ ($A^H = \bar{A}^T$) to show that $\| A \| \leq \sqrt{\max_j \lambda_j}$.

(d) Show that in fact $\| A \| = \sqrt{\max_j \lambda_j}$.

Problem 2. Let A be an $m \times n$ matrix where $m > n$. We are particularly interested in the case when A does not have full rank. Suppose $b \in \mathbb{R}^m$ is a known vector. Consider the family of functions

$$\psi_\alpha(x) = ||b - Ax||_2^2 + \alpha ||x||_2^2$$

where $\alpha > 0$ is a positive number.

(a) Find $\nabla \psi_\alpha$ and use it to derive the equivalent of the normal equations for finding the value(s) of x that minimize(s) ψ_α.

(b) Show that the minimum is attained at a unique vector $x = x_\alpha$.

(c) Let x^* be the solution of the least squares problem

$$\min_{x\in\mathbb{R}^n} ||b - Ax||_2^2$$

of minimum norm. Show that if $\alpha = \alpha_k \to 0$ as $k \to \infty$ then $x_{\alpha_k} \to x^*$.

Problem 3. Suppose A is an $n \times n$ symmetric positive definite matrix with corresponding A-inner product and A-norm defined by

$$\langle x, y\rangle_A = x^T A y \qquad \text{and} \qquad ||x||_A^2 = x^T A x.$$

Consider the system of equations $Ax = b$. Recall, the conjugate gradient method is an iterative method for finding the solution x^* that starts with a vector x_0 and finds iterates x_1, x_2, ... as follows:

$$\begin{aligned} r_i &= b - Ax_i \\ p_i &= r_i - \sum_{k<i} \frac{p_k^T A r_i}{||p_k||_A^2} p_k \\ x_{i+1} &= x_i + \frac{p_i^T r_i}{||p_i||_A^2} p_i \end{aligned}$$

Notice the definition of the p_i's uses Gram-Schmidt to ensure they are mutually A-orthogonal. For simplicity, you may assume in this problem that $r_i \neq 0$ for $i = 0, 1, \ldots, n-1$.

(a) Let x^* be the solution to $Ax = b$. Show

$$x^* = \sum_{i=0}^{n-1} \left(\frac{p_i^T b}{||p_i||_A^2} \right) p_i.$$

(b) Writing $x_0 = \sum_{i=0}^{n-1} \beta_i p_i$, show

$$x_k = \sum_{i=0}^{k-1} \left(\frac{p_i^T b}{||p_i||_A^2} \right) p_i + \sum_{i=k}^{n-1} \beta_i p_i$$

for $k = 1, 2, \ldots, n$.

(c) Use parts (a) and (b) to show x_k minimizes the function $\psi(x) = ||x - x^*||_A^2$ in the affine space

$$S_{k-1} = x_0 + \text{Span}\{p_0, p_1, \ldots, p_{k-1}\}$$

and deduce that r_k is orthogonal to $p_0, p_1, \ldots, p_{k-1}$ (with respect to the regular inner product).

(d) Show

$$\text{Span}\{r_0, r_1, \ldots, r_k\} = \text{Span}\{p_0, p_1, \ldots, p_k\} = \text{Span}\{r_0, Ar_0, \ldots, A^k r_0\}$$

for $k = 0, 1, \ldots, n-1$.

(e) Show that for each $k = 2, 3, \ldots, n-1$, r_k is A-orthogonal to p_i for $i \leq k-2$. What is the significance of this in terms of the execution of the conjugate gradient method?

Problem 4.

(a) Let $A = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$ be the $n \times n$ matrix A whose entries are all ones. Find all the eigenvalues and a full set of orthogonal eigenvectors of A. Make sure you explain how you got your answers. Hint: You might not want to use the determinant.

(b) One of the difficulties in calculating eigenvalues and eigenvectors is their sensitivity with respect to the entries in the matrix. For $\varepsilon \in \mathbb{R}$ Consider the following $n \times n$ matrix A_ε: ones on the first sub-diagonal, ε (a generally small nonzero number) in the upper right hand corner and zeros elsewhere. For example, when $n = 3$ the matrix A_ε is shown here: $A_\varepsilon = \begin{bmatrix} 0 & 0 & \varepsilon \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. What is the effect on the eigenvalues and eigenvectors of introducing this small nonzero entry ε into the matrix A_0 (i.e. the matrix A_ε with $\varepsilon = 0$).