ODE EXAM - Spring 2020

The exam has **four** problems. Each problem is worth 10 points. Do all four problems.

1. Let $\varepsilon \in [-1, 1]$ be a constant, and consider the differential equation

$$y'(t) = -y(t) + \varepsilon \sin\big(y(t)\big), \ t > 0.$$

   Prove that all solutions of the equation satisfy $\lim_{t \to +\infty} |y(t)| = 0$.

2. Consider the linear system of ODEs in $\mathbb{R}^2$,

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \ln(1+|t|) & \frac{1}{1+t^2} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

   Show that there exists at least one solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

   such that

$$\lim_{t \to +\infty} \Big(|x(t)| + |y(t)| \Big) = +\infty.$$

3. For the nonlinear system

$$\begin{cases} x' = -y - (x^2+y^2)x \\ y' = \quad x - (x^2+y^2)y, \end{cases}$$

   determine the location and type of all critical points.

4. Let σ, r and b be positive constants, and consider the Lorentz system

$$\begin{cases} x' = -\sigma(x-y) \\ y' = rx - y - xz \\ z' = -bz + xy. \end{cases}$$

   (a) Prove that if $r > 1$, then the origin $(0, 0, 0)$ is a hyperbolic fixed point of saddle type.

   (b) Describe the tangent space to the stable manifold of the system at the origin when $b = 2, r = 4, \sigma = 1$.