PARTIAL DIFFERENTIAL EQUATIONS QUALIFYING EXAM

Spring 2020

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Let $u(x, y)$ be a harmonic function on $\mathbb{R}^2$, and suppose that

$$\int\int_{\mathbb{R}^2} |\nabla u(x, y)|^2 \, dxdy < \infty.$$

Show that u is a constant function.

2. Solve the following Cauchy problem

$$\begin{aligned} \partial_t u - x\partial_x u + u - 1 = 0, &\qquad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ u(0, x) = \cos x, &\qquad x \in \mathbb{R} \end{aligned}$$

Discuss the behavior of $|\partial_x u(t, x)|$ for $t \to \infty$.

3. Consider the bounded smooth solution u to the linear heat equation

$$\partial_t u = \Delta u, \quad u(0, x) = f(x)$$

on $[0, +\infty) \times \mathbb{R}^d$., with initial data $f \in L^2(\mathbb{R}^d)$.

(a) Prove that $\|u(t)\|_{L^2(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}$ for $t \geq 0$.

(b) Prove that $\lim_{t \to +\infty} \|u(t)\|_{L^2(\mathbb{R}^d)} = 0$.