1. Suppose for some unknown $\theta_0 \in \mathbb{R}$ we observe pairs (x_i, y_i), with x_i non-random and

$$y_i = \theta_0 x_i + \epsilon_i \quad i = 1, 2, \ldots, n$$

where $\epsilon_1, \ldots, \epsilon_n$ are random errors. We estimate θ by least squares, that is, by the value $\widehat{\theta}_n$ minimizing

$$J_n(\theta) = \frac{1}{2n} \sum_{i=1}^{n} (y_i - \theta x_i)^2 .$$

   (a) Compute $\widehat{\theta}_n$. Are any conditions needed for this estimate to exist?

   (b) Find conditions on the distribution of y_i, as minimal as you can, for which $\widehat{\theta}_n$ will be unbiased for θ_0.

   (c) Under the assumptions in (a) and (b), and that the ϵ_i errors each have variance σ^2, find conditions, as simple as possible, under which $\widehat{\theta}_n$ will be consistent.

   (d) Without specifying the precise conditions, find the limiting asymptotic distribution of $\widehat{\theta}_n$ once it has been properly scaled and centered.

2. (a) Let $Z_1, \ldots, Z_n$ be i.i.d. $N(\mu, \sigma^2)$ random variables, for some $\sigma > 0$ and μ. Define $X_i = e^{Z_i}$, $i = 1, \ldots, n$. Find the mean, variance, and median of the distribution of X_1 in terms of μ and σ.

   (b) Let $M_n = (\prod_{i=1}^{n} X_i)^{1/n}$ be the geometric mean of the X_i. In terms of μ and σ, find

      i. the c.d.f. of M_n

      ii. the mean, variance, and median of M_n (*hint:* use part 2a)

      iii. $\lim_{n\to\infty} M_n$

   (c) Show that, for any $n \geq 1$,

$$\lim_{n'\to\infty} M_{n'} \leq EM_n \leq EX_1.$$