Geometry and Topology Graduate Exam
Spring 2020

Solve as many problems as you can. Partial credit will be given to partial solutions.

Problem 1. Show that the subset

$$X = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5; x_1^4 + x_2^4 = 1 + x_3^2 + x_4^2 + x_5^2\} \subset \mathbb{R}^5$$

is an orientable manifold of dimension 4. Make sure that you justify the word "orientable".

Problem 2. Let M be a compact orientable manifold with boundary. Show that there is no differentiable map $f: M \to \partial M$ such that $f(x) = x$ for every $x \in \partial M$.
Possible hint: Consider a volume form on ∂M.

Problem 3. Let M be an m-dimensional submanifold of $\mathbb{R}^{2m+2}$. Show that there is a hyperplane $H \subset \mathbb{R}^{2m+2}$ such that the restriction to M of the orthogonal projection $\pi_H: \mathbb{R}^n \to H$ is injective.
Possible hint: Use a suitable map $\{(x, y) \in M \times M; x \neq y\} \to S^{2m+1}$.

Problem 4. Consider the space X obtained from the cylinder $S^1 \times [0, 1]$ by identifying the antipodal points of the circle $S^1 \times \{0\}$ and identifying the antipodal points of the circle $S^1 \times \{1\}$. (Recall that the antipodal point of $(x, y) \in S^1 \subset \mathbb{R}^2$ is the point $(-x, -y)$.) Compute the fundamental group of X.

Problem 5. Use the Mayer–Vietoris exact sequence to compute, for any space X, the homology groups $H_p(X \times S^n)$ in terms of the homology groups $H_q(X)$ (for any coefficients).

Problem 6. Let $X = S^1 \vee S^1$ be the wedge sum of two circles, namely the union of two circles meeting in exactly one point x_0. Let a_1, a_2 be the generators of $\pi_1(X; x_0)$ represented by loops going around the first and second circles, respectively. By covering space theory, for every subgroup H of $\pi_1(X; x_0)$ there is a connected covering space $p: \widetilde{X} \to X$ such that the image of $p_*: \pi_1(\widetilde{X}; \widetilde{x}_0) \to \pi_1(X; x_0)$ is equal to H. Draw a picture of $\widetilde{X}$ when H is the subgroup generated by the element $a_1 a_2 a_1 a_2 \in \pi_1(X; x_0)$.

Problem 7. For the n-dimensional sphere S^n, consider the inclusion map $i: S^n \to \mathbb{R}^{n+1} - \{0\}$ and a closed differential form $\omega \in \Omega^n(\mathbb{R}^{n+1} - \{0\})$. Let M be a compact oriented n-dimensional manifold without boundary and, for a differentiable map $f: M \to \mathbb{R}^{n+1} - \{0\}$, let $f^*(\omega) = \Omega^n(f)(\omega) \in \Omega^n(M)$ denote the pullback of ω under f. Show that the integral $\int_M f^*(\omega)$ is an integer multiple of $\int_{S^n} i^*(\omega)$.
Possible hint: Consider the radial projection $p: \mathbb{R}^{n+1} - \{0\} \to S^n$.