REAL ANALYSIS GRADUATE EXAM

Spring 2020

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be strictly increasing and continuous. True or false: If $A \subseteq \mathbb{R}$ is Lebesgue measurable then $f^{-1}(A)$ is Lebesgue measurable. Provide a detailed justification of your answer.

2. Prove that the limit

$$\lim_{n\to\infty} n \int_{1/n}^{1} \frac{\cos(x+1/n) - \cos x}{x}\, dx$$

exists. (Above, consider $n \in \mathbb{N}$.)

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $M > 0$. Prove that the following two conditions are equivalent:
(a) $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$,
(b) f is absolutely continuous function which satisfies $|f'(x)| \le M$ for all $x \in \mathbb{R}$.

4. Let m be the Lebesgue measure on $\mathbb{R}$ and m^* the Lebesgue outer measure. For $A \subseteq \mathbb{R}$, define

$$m_*(A) = \sup\{m(K) : K \subseteq A, K \text{ compact}\}.$$

Prove that if $m^*(A) = m_*(A)$ and $m^*(A) < \infty$, then A is Lebesgue measurable.