COMPLEX ANALYSIS GRADUATE EXAM

Spring 2020

Answer all four questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning, and guesswork will lower your score. Start each problem on a fresh sheet of paper, and write on only one side of the paper.

1. Compute

$$\int_0^\infty \frac{\cos x \, dx}{(1+x^2)^2}.$$

2. Let $\Omega \subseteq \mathbb{C}$ be an open set such that $\{|z| \leq 1\} \subseteq \Omega$. Let $f_n \colon \Omega \to \mathbb{C}$ be a sequence of analytic functions which converge uniformly on compact subsets of Ω to $f \colon \Omega \to \mathbb{C}$. Assume $f(z) \neq 0$ for all $|z| = 1$. Prove that there exists $N \in \mathbb{N}$ so that f_n and f have the same number of zeroes in the unit disk $\{|z| < 1\}$ for all $n \geq N$.

3. Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. Let $f \colon \mathbb{D} \to \mathbb{D}$ be analytic and not identically z. Prove that f has at most one fixed point in $\mathbb{D}$. Is the same true if $\mathbb{D}$ is replaced by a simply connected bounded subset of $\mathbb{C}$? Prove or provide a counterexample.

4. Let f be an non-constant entire function. Prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$.