Algebra Qualifying Exam - January 2020

1. Classify all groups of order 75 up to isomorphism.
2. Let G be a group acting transitively on a set X of size $n > 1$.
   (a) If G is finite, show that there exists $g \in G$ so that $gx \neq x$ for all $x \in X$ (hint: count the number of g such that $gx = x$ for some $x \in X$).
   (b) Give an example to show this can fail for G infinite (Hint: consider $GL_n(\mathbb{C})$ with X the set of 1-dimensional subspaces of the space of column vectors).
3. Let R be an integral domain with quotient field F.
   (a) If M is a maximal ideal of R, show that the localization R_M of R at M naturally embeds in F.
   (b) Show that $R = \cap_M R_M$ where the intersection is over all maximal ideals (hint: If $s \in \cap R_M$, let $I = \{r \in R | rs \in R\}$. show that I is an ideal and is not contained in any maximal ideal M).
4. Let K be a field of characteristic 0 containing all mth roots of unity. Let L/K be a field extension and $a \in L$ such that $a^m \in K$. Prove that $K(a)/K$ is Galois with a Galois group that is cyclic of order dividing m.
5. Let k be field with $f, g \in k[x_1, \ldots, x_n]$. Show that $f(a_1, \ldots, a_n) = 0$ if and only if $g(a_1, \ldots, a_n) = 0$ is equivalent to f and g having exactly the same (monic) irreducible factors.
6. Assume that R is a semisimple ring which is a finite-dimensional algebra over a field k , such that for every $r \in R$, there exists a positive integer $n = n(r)$ such that $r^n \in Z(R)$ the center of R. Prove that R is commutative in the following two cases:
   (a) k is finite;
   (b) $k = \mathbb{R}$. (hint: first show that there exists $x \in \mathbb{C}$ such that $x^n \notin \mathbb{R}$, for all positive n)