MATH 507a **QUALIFYING EXAM** **January 2020**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Let $\{X_n, n \geq 1\}$ be iid uniform in [0,1]. For which values of $\gamma > 0$ does the series $\sum_n (X_n + n^{-\gamma})^{n^{\gamma+1}}$ converge a.s.?

HINT: What happens when $X_n + n^{-\gamma} \geq 1$? Consider also events $X_n + n^{-\gamma} < 1 - \epsilon_n$ for some natural choice of ϵ_n.

(2) Let $\{X_i\}$ be iid nonnegative random variables with mean μ and variance $\sigma^2 \in (0, \infty)$. Let $S_n = \sum_{i=1}^n X_i$. Show that

$$\sqrt{S_n} - \sqrt{n\mu} \implies N\left(0, \frac{\sigma^2}{4\mu}\right) \quad \text{in distriubution, as } n \to \infty.$$

HINT: For $T_n = S_n - n\mu$ we have $S_n = n\mu\left(1 + \frac{T_n}{n\mu}\right)$. Also, you may assume Slutsky's Theorem: if $U_n \implies U$ in distribution and $V_n \to 0$ in probability, then $U_n + V_n \implies U$.

(3)(a) Suppose $\varphi(t)$ is a characteristic function of some random variable. Show that Re $\varphi(t)$ is also a characteristic function.

(b) Suppose X_1, X_2 are iid. Show that the distribution of $X_1 - X_2$ cannot be the uniform distribution over any interval $[a, b]$. HINT: This doesn't use part (a), but part (a) may get you thinking in a useful direction.