MATH 505a **QUALIFYING EXAM** **January 2020**

Answer all three questions. Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Unacknowledged omissions, incorrect reasoning and guesswork will lower your score. Start each problem on a new page and write on only one side of the paper. For problems with multiple parts, if you cannot get an answer to one part, you might still get credit for other parts by assuming the correct answer to the part you could not solve. Be aware of the passage of time, so that you can attempt all three problems.

(1) Each pack of bubble gum contains one of n types of coupon, equally likely to be each of the types, independently from one pack to another. Let T_j be number of packs you must buy to obtain coupons of j different types. Note that $T_1 = 1$ always.

(a) Find the distribution and expected value of $T_2 - T_1$ and of $T_3 - T_2$.

(b) Compute $\mathbb{E}\, T_n$.

(c) Fix k and let A_i be the event that none of the first k packs you buy contain coupon i. Find $P(A_1 \cup A_2 \cup A_3 \cup A_4)$. Then fix $\alpha > 0$, take $k = \lfloor \alpha n \rfloor$ and find the limit of this probability as $n \to \infty$. Here $\lfloor x \rfloor$ denotes the largest integer $\leq x$. HINT: Consider probabilities $P(A_i), P(A_i \cap A_j)$, etc.

(d) Assume there are $n = 4$ coupon types; find $P(T_4 > k)$ for all $k \geq 4$. HINT: This is short if you use what you've already done.

(2) Let X be exponential(λ) (that is, density $f(x) = \lambda e^{-\lambda x}$.) The *integer part* of X is $\lfloor X \rfloor = \max\{k \in \mathbb{N} : k \leq X\}$. The *fractional part* of X is $X - \lfloor X \rfloor$. Show that $\lfloor X \rfloor$ and $X - \lfloor X \rfloor$ are independent.

HINT: In general, two random variables U, V are independent if the distribution of V conditioned on $U = u$ doesn't depend on u.

(3) Let X_1, X_2, X_3 be i.i.d. uniform in $[0, 1]$. Let $X_{(1)}$ be the smallest of the 3 values, $X_{(2)}$ the second smallest, and $X_{(3)}$ the largest.

(a) Find the distribution function and the expected value for $X_{(1)}$.

(b) Find the distribution function and the density of $X_{(2)}$.